Questions

1. Orthonormality

(a) Suppose you have a real orthonormal matrix $U$, i.e., the columns of $U$ are orthonormal, and you have real vectors $\vec{x}_1, \vec{x}_2, \vec{y}_1, \vec{y}_2$ such that

$$\vec{y}_1 = U \vec{x}_1$$
$$\vec{y}_2 = U \vec{x}_2$$

Calculate $\langle \vec{y}_1, \vec{y}_2 \rangle = \vec{y}_1^\top \vec{y}_2$ in terms of $\langle \vec{x}_1, \vec{x}_2 \rangle = \vec{x}_2^\top \vec{x}_1 = \vec{x}_1^\top \vec{x}_2$.

(b) Following the previous question, express $\|\vec{y}_1\|^2_2$ and $\|\vec{y}_2\|^2_2$ in terms of $\|\vec{x}_1\|^2_2$ and $\|\vec{x}_2\|^2_2$.

(c) Suppose you observe data coming from the model $y = \vec{a}^\top \vec{x}$. Suppose you have $m$ data points $(\vec{x}_i, \vec{y}_i)$. Set up a least squares formulation for estimating $\vec{a}$ and find the solution to the least squares.

(d) Now suppose $V$ is an orthonormal square matrix and we observe data points transformed by $V^\top$. In other words, we have observations

$$\vec{\tilde{x}} = V^\top \vec{x}$$

with the $y$’s unchanged. Recall that originally we had $y \approx \vec{a}^\top \vec{x}$. Suppose we want to write the least squares in terms of $\vec{\tilde{x}}$ as $y \approx \vec{\tilde{a}}^\top \vec{\tilde{x}}$. Set up a least squares formulation for estimating $\vec{\tilde{a}}$ from the $(\vec{\tilde{x}}, y)$ pairs and find the solution to the least squares.
(e) Now suppose that we have the matrix
\[
\begin{bmatrix}
\vec{x}_1^T \\
\vec{x}_2^T \\
\vdots \\
\vec{x}_m^T
\end{bmatrix}
:= X = U\Sigma V^T.
\]
and suppose that the orthonormal matrix in the previous part is the $V$ corresponding to this SVD representation. **Set up a least squares formulation for estimating $\vec{a}$ and find the solution to the least squares. Is there anything interesting going on?**

2. Geometric interpretation of the SVD

In this exercise, we explore the geometric interpretation of symmetric matrices and how this connects to the SVD. We consider how a real $2 \times 2$ matrix acts on the unit circle, transforming it into an ellipse. It turns out that the principal semiaxes of the resulting ellipse are related to the singular values of the matrix, as well as the vectors in the SVD.

(a) Consider the real $2 \times 2$ matrix
\[
A = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}.
\]
Now consider the unit circle in $\mathbb{R}^2$,
\[
S = \left\{ \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \mid 0 \leq \theta < 2\pi \right\}.
\]
Plot $AS$ on the $\mathbb{R}^2$ plane.

(b) We can think of the unit circle as solutions to $\vec{x}^T\vec{x} = 1$ for a two-dimensional vector $\vec{x}$. **Verify that the unit-circle satisfies this equation. Then, find a diagonal matrix $W$ such that $(AS)^TW(AS) = 1$.** Hint: The SVD of $A$ is
\[
A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
(c) Consider the columns of the matrices $U, V$ obtained in the previous part, and treat them as vectors in $\mathbb{R}^2$. Let $U = (\vec{u}_1, \vec{u}_2)$, $V = (\vec{v}_1, \vec{v}_2)$. Let $\sigma_1, \sigma_2$ be the singular values of $A$, where $\sigma_1 \geq \sigma_2$.

Draw in your plot of $AS$ the vectors $\sigma_1 \vec{u}_1$ and $\sigma_2 \vec{u}_2$, drawn from the origin. What do these vectors correspond to geometrically?

(d) Repeat what you did above for the matrix $A = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$.

Do you notice something?

Hint: The SVD of $A$ is

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(e) Consider the case where $A$ is a real $n \times n$ symmetric matrix. What do you observe geometrically in this case?

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