1 Transfer Functions

At this point, you are now more or less familiar with the basic tools of frequency-domain circuit analysis. This is a fancy-sounding word for something that is quite simple. In the “frequency domain,” we represent voltages and currents using complex-valued phasors, and we represent capacitors and inductors with their complex-valued impedances. As we saw in lecture and the previous discussion, although each real sinusoidal input is mathematically the superposition of one complex exponential $e^{+j\omega t}$ and another complex exponential $e^{-j\omega t}$, because of the properties of complex conjugacy, it suffices to just keep track of what happens to the exponential $e^{+j\omega t}$. To understand what happens to the other term, we just take a complex conjugate. This means that looking at $s = +j\omega$ suffices.

As a result, frequency-domain analysis avoids having to deal with differential equations: all equations that need to be solved are algebraic, just like standard circuit analysis from 16A. However, because the (interesting) circuits have capacitors and inductors in them, we need to have a way to think about the frequency-dependent behavior. After all, the impedances corresponding to a capacitor and inductor are frequency dependent.

To do this, we have a pair of tools — one is analytic in terms of mathematical expressions, and the other is graphical in terms of plots. These two very powerful tools for describing the input-output behavior of a circuit are called the transfer function and the Bode plot.

Let’s consider an example of why transfer functions are useful. Take a look at the following circuit:

![Figure 1: First order RC low-pass filter as a circuit. Because it is a circuit, it is naturally thought of as existing in the time domain. That is, it is going to be described by differential equations.](image)

Given some input $v_{\text{in}}(t)$, we would like to describe the resulting output $v_{\text{out}}(t)$. We could use transient-style analysis, we would have to solve the differential equation

$$\frac{d}{dt}v_{\text{out}}(t) + \frac{1}{RC}v_{\text{out}}(t) - \frac{1}{RC}v_{\text{in}}(t) = 0,$$  \hspace{1cm} (1)

and match to some initial conditions for every different input $v_{\text{in}}(t)$. Then we might have to do integration by parts to get an analytic solution. Though very reliable and mechanical, this procedure is involved and can
feel repetitive. Not only can one make mechanical calculation mistakes, but it also is not very friendly for design intuition.

On the other hand, in the frequency domain, we would represent the circuit using impedances like this:

\[ \tilde{V}_{\text{in}} \quad R \quad \tilde{I} \quad \tilde{V}_{\text{out}} \]

\[ + \quad - \]

\[ Z_C \]

Figure 2: The same first order RC low-pass filter in the frequency domain. This looks like a 16A-style circuit and is clearly some kind of voltage divider.

Here, \( \tilde{V}_{\text{in}} \) is a given voltage phasor with frequency \( \omega \). \( \tilde{V}_{\text{out}} \) is in the middle of a voltage divider with the resistor \( R \) and the impedance \( Z_C = \frac{1}{j\omega C} \), and we can write

\[ \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{1}{1 + j\omega RC} \]  \( \tilde{V}_{\text{in}} \).

While this analysis is only valid for sinusoidal-type inputs, it is a lot more intuitive and design-friendly. Defining \( H(\omega) \) by

\[ H(\omega) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{1}{1 + j\omega RC}, \]

we can rewrite the input-output relationship in the phasor domain as \( \tilde{V}_{\text{out}} = H(\omega)\tilde{V}_{\text{in}} \). \( H(\omega) \) is a simple function relating the input and output of the circuit. Even if \( \tilde{V}_{\text{in}} \) were another sinusoid, we could use the same phasor-domain formula to compute \( \tilde{V}_{\text{out}} \).

The function \( H \) is called a transfer function. A transfer function is a ratio between two phasors in a circuit — usually an input, which is known, and an output, which is to be determined. The transfer function tells us how the known input is “transferred” to the output. The purpose of transfer functions is to facilitate design.

The transfer function of a circuit describes how the circuit responds to inputs of different frequencies. For any choice of \( \omega \), e.g. \( \omega = 1000 \) radians/s, \( H(\omega) \) predicts its effect on the circuit’s output. Additionally, \( H(\omega) \) can be analyzed for asymptotically “low” or generally “high” frequencies, by taking the limit of \( H(\omega) \) as \( \omega \to 0 \) or \( \omega \to \infty \), respectively.

Let’s try this on the example we just did. At low frequencies,

\[ \lim_{\omega \to 0} H(\omega) = \frac{1}{1} = 1. \]

This means that, at low frequencies, we have \( \tilde{V}_{\text{out}} \approx \tilde{V}_{\text{in}} \). On the other hand, at high frequencies,

\[ \lim_{\omega \to \infty} H(\omega) = \lim_{\omega \to \infty} \frac{1}{j\omega RC} = 0. \]

So at high frequencies, we have \( \tilde{V}_{\text{out}} \approx 0 \). In other words, this circuit lets low-frequency inputs pass through the circuit almost unaffected but impedes high-frequency inputs from passing. For this reason, we refer to this type of circuit as a low-pass filter.
2 Plotting Transfer Functions as Bode Plots

It is often useful to be able to plot the transfer function of a circuit as a function of frequency. This allows us to visualize how our circuit reacts to any particular frequency. Because the frequency response is a complex-valued function, we have to find a natural way to plot this. Because we want to do design, we want to plot in a way that makes it easiest to compose transfer functions. Because a transfer function is fundamentally a multiplicative object, we would rather plot in a way that turns multiplication into addition. (Because then we can reason about what happens when we compose transfer functions by just adding the plots together.) This means that some kind of logarithmic scaling is useful for the transfer function itself. Consequently, it turns out to be useful to plot the magnitude and phase of the frequency response separately. The phase is naturally logarithmic (because it lives in the exponent of $e^{j\theta}$) and so we plot the phase on a linear scale. The magnitude is not naturally logarithmic, and so we plot the log of the magnitude.

This tells us how the vertical axes should work for both plots. But what about the horizontal axis that represents the frequency $\omega$? Here, we just remember that there are many orders of magnitude that are spanned by frequencies of interest in real-world design — from the 60Hz of wall power to the kHz of many natural signals of interest to the MHz and GHz of interfering signals in the built environment. Only a logarithmic scale has any hope of showing us that kind of dynamic range of frequencies. So the horizontal axis is generally going to be logarithmic, unless we are zooming in on some particular narrow frequency range of interest.

To show you how it works, we'll plot the transfer function for the RC low-pass filter we’ve been looking at. We need specific values to make the plot, so we’ll take $R = 1\,k\Omega, C = 1\,\mu F$. We’ve analyzed this circuit before and found the transfer function to be:

$$H(\omega) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{1}{1 + j\omega RC}.$$  \hspace{1cm} (6)

Since $H(\omega)$ is complex, we need two separate plots to show the whole thing. We could either plot the real part and the imaginary part, or the magnitude and phase. It will be more useful to plot the magnitude and the phase, since that will make the effect of $H(\omega)$ on phasors more apparent. First, we’ll find the magnitude and phase response from the transfer function:

$$|H(\omega)| = \left| \frac{1}{j\omega RC + 1} \right| = \frac{|1|}{|j\omega RC + 1|} = \frac{1}{\sqrt{(\omega RC)^2 + 1}},$$

$$\angle H(\omega) = \angle \frac{1}{j\omega RC + 1} = \angle 1 - \angle (j\omega RC + 1) = 0 - \text{atan2}(\omega RC, 1) = -\text{atan2}(\omega RC, 1).$$

Just to give you a mental picture, using plotting software to plot the exact $|H(\omega)|$ and $\angle H(\omega)$ gives the following:
For now, we will focus on the plot of the magnitude of the transfer function, $|H(\omega)|$. This plot verifies our limit-based analysis of $H(\omega)$: The magnitude stays near $10^0$ at low frequencies, then at some point it starts to decrease. The point at which $|H(\omega)|$ starts to decrease is called the cutoff frequency of the filter, and denoted $\omega_c$. These curves are generally curvy and have no sharp point of transition. Consequently, $\omega_c$ is usually chosen to be around where $|H(\omega)| = \frac{1}{\sqrt{2}}$.

The astute reader may have noticed that these plots are in the log-log and semilog coordinate systems, which make these functions easier to visualize. Linear-scale plots are included below for comparison. In the future we will rely on log scales to approximate Bode plots.
1. Transfer function practice

Now that you’ve seen how to derive a transfer function for a simple circuit, you’ll be deriving some transfer functions on your own. To start off with, determine \( H(\omega) = \frac{\tilde{V}_{\text{out}}(\omega)}{\tilde{V}_{\text{in}}(\omega)} \) for the following circuits, and use the transfer function you find to describe how the circuit responds to low and high frequencies.

(a) Determine \( H(\omega) = \frac{\tilde{V}_{\text{out}}(\omega)}{\tilde{V}_{\text{in}}(\omega)} \). How does this circuit respond as \( \omega \to 0 \) (low frequencies)? as \( \omega \to \infty \) (high frequencies)?

\[
\tilde{V}_{\text{in}}(\omega) \quad + \quad C \quad - \quad R \quad \tilde{V}_{\text{out}}(\omega)
\]

**Answer:** Just like the simple RC example in the text, we’ll use the voltage divider formula to find \( \tilde{V}_{\text{out}} \):

\[
\tilde{V}_{\text{out}} = \frac{Z_R}{Z_R + Z_C} \tilde{V}_{\text{in}}.
\]

A little algebraic manipulation gives

\[
H(\omega) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega}{j\omega + \frac{1}{RC}}.
\]  

(7)

At low frequencies, we have

\[
\lim_{\omega \to 0} H(\omega) = 0,
\]  

(8)

while at high frequencies, we have

\[
\lim_{\omega \to \infty} H(\omega) = 1.
\]  

(9)

So this circuit is a **high-pass filter**.
(b) Determine \( H(\omega) = \frac{V_{\text{out}}(\omega)}{V_{\text{in}}(\omega)} \). How does this circuit respond as \( \omega \to 0 \) (low frequencies)? as \( \omega \to \infty \) (high frequencies)?

Answer: The strategy is the same as the previous part, that is, using the voltage divider formula, \( i.e., \)

\[
\tilde{V}_{\text{out}} = \frac{Z_R}{Z_R + Z_L} \tilde{V}_{\text{in}}.
\]

A similar manipulation to the previous part gives

\[
H(\omega) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega L R}.
\]

(10)

At low frequencies, we have

\[
\lim_{\omega \to 0} H(\omega) = 1,
\]

(11)

while at high frequencies, we have

\[
\lim_{\omega \to \infty} H(\omega) = 0.
\]

(12)

So this circuit is a low-pass filter. Notice that this circuit resembles the one in the previous part, except we have replaced the capacitor with an inductor. The effect of this change was to reverse the low-frequency and high-frequency behavior of the circuit! Another example of the complementarity of capacitors and inductors.
(c) Determine $H(\omega) = \frac{V_{\text{out}}(\omega)}{V_{\text{in}}(\omega)}$. How does this circuit respond as $\omega \to 0$ (low frequencies) as $\omega \to \infty$ (high frequencies)?

**Answer:** Even though there are three components instead of two, we can still use the voltage divider formula by treating $R_2$ and $C$ as a single impedance $Z = R_2 + \frac{1}{j\omega C}$. This would give us

$$
\tilde{V}_{\text{out}} = \frac{Z}{ZR_1 + Z} \tilde{V}_{\text{in}}.
$$

Then, the transfer function is

$$
H(\omega) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{R_2 + \frac{1}{j\omega C}}{R_1 + R_2 + j \frac{1}{\omega C}} = \frac{1 + j\omega R_2 C}{1 + j\omega C(R_1 + R_2)}.
$$

(13)

At low frequencies, we have

$$
\lim_{\omega \to 0} H(\omega) = 1,
$$

(14)

while at high frequencies, we have

$$
\lim_{\omega \to \infty} H(\omega) = \frac{R_2}{R_1 + R_2}.
$$

(15)

So at high frequencies, this circuit behaves like a regular voltage divider with just $R_1$ and $R_2$, as if the capacitor had vanished. This circuit is like a combination of a high-pass filter and a voltage divider: low frequency inputs are removed, and high-frequency signals are diminished.
2. Bode plot practice

For this problem, we will try to follow the same reasoning from section 2 to construct a Bode plot of the following circuit:

Here is a blank log-log plot for you to draw your Bode plot in.

Bode plot of $|H(\omega)|$ (for you to draw).

(a) Write an expression for $\log_{10}(|H(\omega)|)$. For now, you can keep it in terms of $R$ and $L$.

Answer: We already found the transfer function in the previous exercise: specifically,

$$H(\omega) = \frac{\bar{V}_{\text{out}}}{\bar{V}_{\text{in}}} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega \frac{L}{R}}.$$  \hspace{1cm} (16)

The magnitude is

$$|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2 \frac{L^2}{R^2}}},$$  \hspace{1cm} (17)

which you can find by taking the square root of $\overline{H(\omega)}H(\omega)$. Taking the logarithm gives

$$\log_{10}(|H(\omega)|) = -\frac{1}{2} \log_{10} \left( 1 + \omega^2 \frac{L^2}{R^2} \right).$$  \hspace{1cm} (18)
(b) What is the cutoff frequency for this circuit? Mark it on the log-log plot with a vertical line.

**Answer:** In this case, it will be the inverse of the LR time constant, that is

\[
\omega_c = \frac{R}{L}.
\]  

(19)

For our given values, that’s

\[
\omega_c = \frac{100 \ \Omega}{100 \ \mu H} = 10^6 \text{ radians/s}.
\]  

(20)

(c) Pick a few representative points (pick some above the cutoff frequency and some below the cutoff frequency) and use a computer/calculator to evaluate the transfer function there. Sketch coarsely the plot of the magnitude of this transfer function.

**Answer:** The final drawing should look like this:

![Bode plot of |H(\omega)|](image-url)
(d) Now suppose we want to compose the filter from the introduction (simple RC low pass filter) with this new circuit. We can compose two circuits by connecting the output of the first circuit into the second circuit, through a unity gain buffer. For this problem, call the circuit from this system with an inductor $H_1$, and the other filter $H_2$. The output of the composed circuit is:

$$H(w) = H_1(w) \cdot H_2(w)$$

i. Draw this circuit.

Answer:
ii. Plot the magnitude of the composed circuit. Below is a log-log plot with the magnitudes of $|H_1(\omega)|$ and $|H_2(\omega)|$ drawn to assist you.

Bode plot approximation of $|H(\omega)|$ (for **you** to draw).

**Answer:** The final magnitude plot should be obtained by adding the two lines from the two transfer functions!
Solution plot of transfer function magnitude

\[ |H(\omega)| \]

\( \omega \)

\( 10^1 \quad 10^2 \quad 10^3 \quad 10^4 \quad 10^5 \quad 10^6 \quad 10^7 \quad 10^8 \)
ii. Plot the phase of the composed circuit. Below is a semi-log plot with the phases $\angle H_1(\omega)$ and $\angle H_2(\omega)$ drawn to assist you.

**Answer:** The final phase plot should be obtained by adding the two lines from the two transfer functions!:
Solution plot of transfer function phase

\[ \angle H(\omega) \]

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