1 Notes

1.1 Discrete Fourier Transform

Assume we are working with an $N$ length discrete signal and we would like to find its discrete frequencies. This is done through the Discrete Fourier Transform (DFT), which is simply a change of basis to what is called the DFT basis.

First, let us vectorize our signal. If $x[n]$ is our input signal, we model it as a vector by letting the $n^{th}$ coordinate be $x[n]$. In other words,

$$\vec{x} = [x[0], x[1], x[2], \ldots, x[N-1]]^T$$

In order to decompose $\vec{x}$ into its constituent frequencies, we must find the vector representation of these frequencies.

Given that we have an $N$ length signal, we have $N$ different discrete frequencies of the following form.

$$u_k[n] = \frac{1}{\sqrt{N}} e^{j \frac{2\pi}{N} kn} \text{ for } k = 0, 1, \ldots, N - 1$$

To simplify we let

$$W_N = e^{j \frac{2\pi}{N}}$$

and we rewrite

$$u_k[n] = \frac{1}{\sqrt{N}} W_N^{kn} \text{ for } k = 0, 1, \ldots, N - 1$$

In building up a frequency basis, we vectorize the above frequencies in a manner similar to how we vectorized $\vec{x}$. Define $\vec{u}_k$ as follows.

$$\vec{u}_k = \frac{1}{\sqrt{N}} \left[ 1, W_N^k, W_N^{k(2)}, \ldots, W_N^{k(N-1)} \right]^T$$

$\{\vec{u}_k\}_{k=0}^{N-1}$ is an orthonormal set of vectors. Recall that an orthonormal set of vectors satisfies the following:

$$\langle \vec{u}_p, \vec{u}_q \rangle = \sum_{n=0}^{N-1} \vec{u}_p[n] \vec{u}_q[n] = \begin{cases} 0, & p \neq q \\ 1, & p = q \end{cases}$$

To see why this set is orthonormal, first consider arbitrary $\vec{u}_p$ and $\vec{u}_q$ such that $p \neq q$. 
\[ \langle \vec{u}_p, \vec{u}_q \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j \frac{2\pi}{N} pn} e^{j \frac{2\pi}{N} qn} \]
\[ = \frac{1}{N} \sum_{n=0}^{N-1} e^{j \frac{2\pi}{N} (q-p)n} \]

Before we continue, let us remind ourselves about the sum of a finite geometric series. Let \( S \) be the sum of the series. Then,

\[ S = 1 + a + a^2 + \cdots + a^{N-1} \]

Then,

\[ aS = a + a^2 + a^3 + \cdots + a^N \]

Subtracting the two, we get,

\[ (1 - a)S = 1 - a^N \implies S = \frac{1 - a^N}{1 - a} \]

Applying this, we get,

\[ \langle \vec{u}_p, \vec{u}_q \rangle = \frac{1}{N} \sum_{n=0}^{N-1} \left( e^{j \frac{2\pi}{N} (q-p) n} \right)^n \]
\[ = \frac{1}{N} \left( 1 - a^N \right) \]
\[ = \frac{1}{N} \left( 1 - e^{j \frac{2\pi}{N} (q-p) N} \right) \]
\[ = \frac{1}{N} \left( 1 - e^{j \frac{2\pi}{N} (q-p)} \right) \]

Note that \( q - p \) is an non-zero integer. This means that,

\[ e^{j \frac{2\pi}{N} (q-p) N} = e^{j 2\pi (q-p)} = 1 \]

Applying this, we get,

\[ \langle \vec{u}_p, \vec{u}_q \rangle = 0 \]

Finally, we also observe that, for a particular DFT basis vector,

\[ \langle \vec{u}_p, \vec{u}_p \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j \frac{2\pi}{N} pn} e^{j \frac{2\pi}{N} pn} = \frac{1}{N} \sum_{n=0}^{N-1} 1 = 1 \]
Thus, \(\{\tilde{u}_k\}_{k=0}^{N-1}\) is an orthonormal set of vectors and is a valid basis. The coefficients of \(\tilde{x}\) within this basis are called the frequency components of \(\tilde{x}\) and are often denoted by \(\tilde{X}\).

\[
\tilde{X} = [(\tilde{u}_0, \tilde{x}), (\tilde{u}_1, \tilde{x}), \ldots, (\tilde{u}_{N-1}, \tilde{x})]^T
\]

The \(k^{th}\) frequency component is the \(k^{th}\) coordinate of \(\tilde{X}\) and is denoted as \(X[k]\). If we want to get the component in the same space as \(\tilde{x}\), we compute the projection.

\[
\text{proj}_{\tilde{u}_k} \tilde{x} = X[k] \tilde{u}_k
\]

It is worthwhile to note that there is a conjugate property we often exploit.

\[
e^{j\frac{2\pi}{N}np} = e^{-j\frac{2\pi}{N}(N-p)n}
\]

This means that,

\[
\tilde{u}_k = \tilde{u}_{N-k}
\]

In fact, since we are using complex exponentials, there is a periodicity that can be succinctly expressed with the remainder operation (also called mod). Let \(p\) be any arbitrary integer.

\[
\tilde{u}_p = \tilde{u}_p \mod N
\]

2. Questions

1. DFT of pure sinusoids

(a) Consider the continuous-time signal \(x(t) = \cos\left(\frac{2\pi}{3} t\right)\). Suppose that we sampled it every 1 second to get (for \(n = 3\) time steps):

\[
\tilde{x} = \left[\cos\left(\frac{2\pi}{3} (0)\right), \cos\left(\frac{2\pi}{3} (1)\right), \cos\left(\frac{2\pi}{3} (2)\right)\right]^T.
\]

Compute \(\tilde{X}\) and the basis vectors \(\tilde{u}_k\) for this signal.

(b) Now for the same signal as before, suppose that we took \(n = 6\) samples. In this case we would have:

\[
\tilde{x} = \left[\cos\left(\frac{2\pi}{3} (0)\right), \cos\left(\frac{2\pi}{3} (1)\right), \cos\left(\frac{2\pi}{3} (2)\right), \cos\left(\frac{2\pi}{3} (3)\right), \cos\left(\frac{2\pi}{3} (4)\right), \cos\left(\frac{2\pi}{3} (5)\right)\right]^T.
\]

Repeat what you did above. What are \(\tilde{X}\) and the basis vectors \(\tilde{u}_k\) for this signal.

(c) Let’s do this more generally. For the signal \(x(t) = \cos\left(\frac{2\pi k}{N} t\right)\), compute \(\tilde{X}\) of its vector form in discrete time, \(\tilde{x}\), of length \(n = N\):

\[
\tilde{x} = \left[\cos\left(\frac{2\pi k}{N} (0)\right), \cos\left(\frac{2\pi k}{N} (1)\right), \ldots, \cos\left(\frac{2\pi k}{N} (N-1)\right)\right]^T.
\]