Notes

Properties of Discrete Time Systems

Consider a discrete-time system with $x[n]$ as input and $y[n]$ as output.

\[ x[n] \rightarrow \square \rightarrow y[n] \]

The following are some of the possible properties that a system can have:

Causality

A causal system has the property that $y[n_0]$ only depends on $x[n]$ for $n \in (-\infty, n_0]$. An intuitive way of interpreting this condition is that the system does not “look ahead.”

Linearity

A linear system has the properties below:

(a) additivity

\[ x_1[n] + x_2[n] \rightarrow \square \rightarrow y_1[n] + y_2[n] \]

(b) scaling

\[ \alpha x[n] \rightarrow \square \rightarrow \alpha y[n] \]

Here, $\alpha$ is some constant.

Together, these two properties are known as **superposition**:

\[ \alpha_1 x_1[n] + \alpha_2 x_2[n] \rightarrow \square \rightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n] \]

Bounded-Input, Bounded-Output (BIBO) Stability

In a BIBO stable system, if $x[n]$ is bounded, then $y[n]$ is also bounded. A signal $a[n]$ is bounded if there exists a $A$ such that $|a[n]| \leq A < \infty \ \forall n$. 
Time invariance

A system is **time-invariant** if its behavior is fixed over time:

\[ x[n - n_0] \rightarrow y[n - n_0] \quad (3) \]

Linear Time Invariant (LTI) Systems

A system is LTI if it is both linear and time invariant. Let \( h[n] \) be the **impulse response** of an LTI system.

That is, \( y[n] = h[n] \) if \( x[n] = \delta[n] \), where \( \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \) is the unit impulse.

An LTI system can be completely characterized by \( h[n] \). The following properties hold:

- An LTI system is causal iff \( h[n] = 0 \forall n < 0 \).
- An LTI system is BIBO stable iff its impulse response is absolutely summable:
  \[ \sum_{n=-\infty}^{\infty} |h[n]| < \infty \]

Convolution Sum

Consider the following LTI system with impulse response \( h[n] \)

\[ x[n] \rightarrow y[n] \]

Notice that we can write \( x[n] \) as a sum of impulses:

\[ x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \]

In addition, we know that:

\[ \delta[n] \rightarrow h[n] \]

By applying the LTI property of our system, we get that

\[ x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \rightarrow y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \]

The expression \( \sum_{m=-\infty}^{\infty} x[m]h[n-m] \) is known as the **convolution sum** and can be written as \( x[n] \ast h[n] \) or \( (x \ast h)[n] \)
Questions

1. Circulant Time-Shift Systems

Imagine we have a system $S_{\rightarrow 2}$ that takes any length 5 input signal and circularly shifts it by two steps. That is, the last two entries roll over to the start and the rest are moved to the right. For example, $S_{\rightarrow 2}([3, 1, 4, 1, 5]) = [1, 5, 3, 1, 4]$.

(a) Is this system linear? That is, for any signals $\vec{x}$ and $\vec{y}$, does $S_{\rightarrow 2}$ fulfill properties (1) and (2)?

(b) What does $S_{\rightarrow 2}$ look like when written as a matrix?

Determine if the following systems are linear, time-invariant, and/or causal.

2. (a) $y[t] = 2x[-2 + 3t] + 2x[2 + 3t]$

(b) $y[t] = 4^t$

(c) $y[t] - y[t - 1] + y[t - 2] = x[t] - x[t - 1] - x[t - 2]$

(d) $y[t] = x[t] + tx[t - 1]$

(e) $y[t] = 2t \cos(x[t])$

3. Convoluted Convolution

Show that convolution is commutative. That is, show that $(x * h)[n] = (h * x)[n]$

4. Mystery System

Consider an LTI system with impulse response

$$h[n] = \frac{1}{2}(\delta[n] + \delta[n - 1])$$

(a) Create a sketch of this impulse response. Is this a finite or infinite impulse response system?

(b) What is the output of our system if the input is the unit step $U[n]$?

(c) What is the output of our system if our input is $x[n] = (-1)^n U[n]$?

(d) This system is called the two-point simple moving average (SMA) filter. Based on the previous parts, why do you think it bears this name?