1 Notes

1.1 Discrete Fourier Transform

Assume we are working with an $N$ length discrete signal and we would like to find its discrete frequencies. This is done through the Discrete Fourier Transform (DFT), which is simply a change of basis to what is called the DFT basis.

First, let us vectorize our signal. If $x[n]$ is our input signal, we model it as a vector by letting the $n^{th}$ coordinate be $x[n]$. In other words,

$$\vec{x} = [x[0], x[1], x[2], \ldots, x[N-1]]^T$$

In order to decompose $\vec{x}$ into its constituent frequencies, we must find the vector representation of these frequencies.

Given that we have an $N$ length signal, we have $N$ different discrete frequencies of the following form.

$$u_k[n] = \frac{1}{\sqrt{N}} e^{j \frac{2\pi}{N} kn} \text{ for } k = 0, 1, \ldots N - 1$$

To simplify we let

$$W_N = e^{j \frac{2\pi}{N}}$$

and we rewrite

$$u_k[n] = \frac{1}{\sqrt{N}} W_N^{kn} \text{ for } k = 0, 1, \ldots N - 1$$

In building up a frequency basis, we vectorize the above frequencies in a manner similar to how we vectorized $\vec{x}$. Define $\vec{u}_k$ as follows.

$$\vec{u}_k = \frac{1}{\sqrt{N}} \left[ 1, W_N^k, W_N^{k(2)}, \ldots, W_N^{k(N-1)} \right]^T$$

$\{\vec{u}_k\}_{k=0}^{N-1}$ is an orthonormal set of vectors. Recall that an orthonormal set of vectors satisfies the following:

$$\langle \vec{u}_p, \vec{u}_q \rangle = \sum_{n=0}^{N-1} \overline{u}_p[n] u_q[n] = \begin{cases} 0, & p \neq q \\ 1, & p = q \end{cases}$$

To see why this set is orthonormal, first consider arbitrary $\vec{u}_p$ and $\vec{u}_q$ such that $p \neq q$. 

\[ \langle \vec{u}_p, \vec{u}_q \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}pn} e^{j\frac{2\pi}{N}qn} \]

Before we continue, let us remind ourselves about the sum of a finite geometric series. Let \( S \) be the sum of the series. Then,

\[ S = 1 + a + a^2 + \cdots + a^{N-1} \]

Then,

\[ aS = a + a^2 + a^3 + \cdots + a^N \]

Subtracting the two, we get,

\[ (1 - a)S = 1 - a^N \implies S = \frac{1 - a^N}{1 - a} \]

Applying this, we get,

\[ \langle \vec{u}_p, \vec{u}_q \rangle = \frac{1}{N} \sum_{n=0}^{N-1} \left( e^{j\frac{2\pi}{N}(q-p)} \right)^n \]

\[ = \frac{1}{N} \left( \frac{1 - a^N}{1 - a} \right) \]

\[ = \frac{1}{N} \left( \frac{1 - e^{j\frac{2\pi}{N}(q-p)N}}{1 - e^{j\frac{2\pi}{N}(q-p)}} \right) \]

Note that \( q - p \) is an non-zero integer. This means that,

\[ e^{j\frac{2\pi}{N}(q-p)N} = e^{j2\pi(q-p)} = 1 \]

Applying this, we get,

\[ \langle \vec{u}_p, \vec{u}_q \rangle = 0 \]

Finally, we also observe that, for a particular DFT basis vector,

\[ \langle \vec{u}_p, \vec{u}_p \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}pn} e^{j\frac{2\pi}{N}pn} = \frac{1}{N} \sum_{n=0}^{N-1} 1 = 1 \]
Thus, \( \{\vec{u}_k\}_{k=0}^{N-1} \) is an orthonormal set of vectors and is a valid basis. The coefficients of \( \vec{x} \) within this basis are called the frequency components of \( \vec{x} \) and are often denoted by \( \vec{X} \).

\[
\vec{X} = [(\vec{u}_0, \vec{x}), (\vec{u}_1, \vec{x}), \ldots, (\vec{u}_{N-1}, \vec{x})]^T
\]

The \( k^{th} \) frequency component is the \( k^{th} \) coordinate of \( \vec{X} \) and is denoted as \( X[k] \). If we want to get the component in the same space as \( \vec{x} \), we compute the projection.

\[
\text{proj}_{\vec{u}_k} \vec{x} = X[k] \vec{u}_k
\]

It is worthwhile to note that there is a conjugate property we often exploit.

\[
e^{j\frac{2\pi}{N}pn} = e^{-j\frac{2\pi}{N}(N-p)n}
\]

This means that,

\[
\vec{u}_k = \overline{\vec{u}_{N-k}}
\]

In fact, since we are using complex exponentials, there is a periodicity that can be succinctly expressed with the remainder operation (also called mod). Let \( p \) be any arbitrary integer.

\[
\vec{u}_p = \vec{u}_p \mod N
\]

2. Questions

1. DFT of pure sinusoids

   (a) Consider the continuous-time signal \( x(t) = \cos\left(\frac{2\pi}{3}t\right) \). Suppose that we sampled it every 1 second to get (for \( n = 3 \) time steps):

\[
\vec{x} = \begin{bmatrix}
\cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right)
\end{bmatrix}^T.
\]

   Compute \( \vec{X} \) and the basis vectors \( \vec{u}_k \) for this signal.

   **Answer:**

\[
\vec{X} = \frac{\sqrt{3}}{2} \begin{bmatrix}
0 & 1 & 1
\end{bmatrix}^T
\]

   **Solution:** Directly apply \( F^* \vec{x} \) to derive \( \vec{X} \).

\[
\vec{X} = \frac{\sqrt{3}}{2} \begin{bmatrix}
0 & 1 & 1
\end{bmatrix}^T
\]

   (b) Now for the same signal as before, suppose that we took \( n = 6 \) samples. In this case we would have:

\[
\vec{x} = \begin{bmatrix}
\cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) & \cos\left(\frac{2\pi}{3}(3)\right) & \cos\left(\frac{2\pi}{3}(4)\right) & \cos\left(\frac{2\pi}{3}(5)\right)
\end{bmatrix}^T.
\]
Repeat what you did above. What are \( \vec{X} \) and the basis vectors \( \vec{u}_k \) for this signal.

**Answer:**
\[
\vec{X} = \frac{\sqrt{6}}{2} \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^T
\]

**Solution:** Directly apply \( F^* \vec{x} \) to derive \( \vec{X} \).
\[
\vec{X} = \frac{\sqrt{6}}{2} \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^T
\]

(c) Let’s do this more generally. For the signal \( x(t) = \cos\left(\frac{2\pi k}{N}t\right) \), compute \( \vec{X} \) of its vector form in discrete time, \( \vec{x} \), of length \( n = N \):
\[
\vec{x} = \begin{bmatrix} \cos\left(\frac{2\pi k}{N}(0)\right) & \cos\left(\frac{2\pi k}{N}(1)\right) & \cdots & \cos\left(\frac{2\pi k}{N}(N-1)\right) \end{bmatrix}^T.
\]

**Answer:**
\[
X[k] = X[N-k] = \frac{\sqrt{N}}{2}
\]
\[
X[m] = 0 \text{ for } m \neq k, N - k.
\]

**Solution:**

i. Show that \( \vec{u}_k + \vec{u}_{N-k} = \frac{2}{\sqrt{N}} \vec{x} \).

ii. Then we have
\[
X[k] = X[N-k] = \frac{\sqrt{N}}{2}
\]
\[
X[m] = 0 \text{ for } m \neq k, N - k.
\]