Notes

Interpolation with Basis Functions

Assume there exists a set of functions $\phi_i(x)$ such that

$$\phi_i(x_i) = 1 \quad \text{and} \quad \phi_i(x_j) = 0 \text{ when } j \neq i$$

We can interpolate between the data points $(x_i, y_i)$ with the function

$$f(x) = \sum_{k=1}^{n} y_k \phi_k(x) \quad \text{because} \quad f(x_i) = \sum_{k=1}^{n} y_k \phi_k(x_i) = y_i$$

We call this set of functions "basis functions".

Sampling theorem

Let $f$ be a signal bandlimited by frequency $\omega_{\text{max}}$, and we sample with a period of $\Delta$ then we can write the sinc-interpolated signal $\hat{f}$

$$\hat{f} = \sum_{n=-\infty}^{\infty} y[n] \Phi(x-n\Delta)$$

Where $\Phi(x) = \text{sinc} \left( \frac{x}{\Delta} \right)$

Then we can recover the signal, i.e. $f = \hat{f}$, if $\omega_{\text{max}} < \frac{\pi}{\Delta}$

Questions

1. Interpolation

Samples from the sinusoid $f(x) = \sin(0.2\pi x)$ are shown in Figure 1. Draw the results of interpolation using each of the following three methods:
Figure 1: Samples of $f(x)$.

(a) Zero order hold interpolation.

**Answer:** Figure 2

(b) Linear interpolation.

**Answer:** Figure 3
2. **Sampling Theorem basics**

Consider the following signal, \( f(x) \) defined as,

\[
f(x) = \cos(2\pi x)
\]

(a) Find the maximum frequency, \( \omega_{\text{max}} \), in radians per second? In Hertz? (From now on, frequencies will refer to radians per second.)

**Answer:** \( \omega_{\text{max}} = 2\pi \) in radians per second, which is 1 Hertz.

(b) What is the smallest sampling \( \Delta \) that would result in an imperfect reconstruction?

**Answer:** From the sampling theorem, we know that \( \Delta \) has an upperbound of \( \frac{\pi}{\omega_{\text{max}}} \) for perfect reconstruction. Hence the smallest \( \Delta \) for which we cannot reconstruct our signal is,

\[
\Delta = \frac{\pi}{2\pi} = \frac{1}{2}
\]
(c) If I sample every $\Delta s$ seconds, what is the sampling frequency?

\[ \omega_s = \frac{2\pi}{\Delta s}. \]

3. More Sampling

Let’s sample the signal from the previous question $f$ with sampling period $\Delta_m = \frac{1}{4}$ s and $\Delta_n = 1$ s and perform sinc interpolation on the resulting samples. Let the reconstructed functions be $g_m$ and $g_n$.

(a) Have we satisfied the Nyquist limit (i.e. the sampling theorem) in any case?

Answer: To satisfy the Nyquist limit, we need the sampling period $\Delta < \frac{1}{T}$. Hence, $\Delta_m$ satisfies Nyquist, but $\Delta_n$ does not.

(b) What is the highest frequency we can reconstruct with the sampling rate $\Delta_n$?

Answer: The sinc functions used to reconstruct $g_n$ are,

\[
\left\{ \text{sinc} \left( \frac{t-k}{1} \right) \right\}_{k \in \mathbb{Z}}.
\]

These functions can represent a maximum frequency of $\pi$.

(c) Based on this answer, can you think of any periodic function that has a frequencies less than or equal to $\pi$ that samples the same as $g_n$?

Answer: Since the frequencies vary from 0 to $\pi$, the smallest period that can be represented is 2. That is to say, functions of period < 2 cannot be captured with the sinc function derived from $\Delta_n$. Since the period must be greater than 2, no sine or cosine function can give the same samples as $g_n$. This means suggests looking into a fairly trivial kind of periodic function: a constant. In particular, the answer to this problem is the constant function that is 1 everywhere.

4. Aliasing

Consider the signal $f(x) = \sin(0.2\pi x)$.

(a) At what period $T$ should we sample so that sinc interpolation recovers a function that is identically zero?

Answer: We want to sample such that our resultant discrete time signal is all zeros. To do this, we can sample at $x = 5k$, for integral values of $k$. Hence, $T = 5$.

(b) At what period $T$ should we sample so that sinc interpolation recovers the function $g(x) = -\sin \left( \frac{\pi}{15} x \right)$?
Answer:  \( T = 7.5 \)

\[
f[n] = \sin(0.2\pi nT)
= \cos\left(0.2\pi nT - \frac{\pi}{2}\right)
= \cos\left(2\pi n - (0.2\pi nT - \frac{\pi}{2})\right)
= \cos\left((2\pi - 0.2\pi T)n + \frac{\pi}{2}\right)
= -\sin((2\pi - 0.2\pi T)n)
= -\sin\left(\frac{\pi}{15}nT\right)
\]

Sampling \( f(x) \)

\( \sin(x) = \cos\left(x - \frac{\pi}{2}\right) \)

\( \cos(x) = \cos(2\pi - x) \)

\( -\sin(x) = \cos\left(x + \frac{\pi}{2}\right) \)

\( 2\pi - 0.2\pi T = \frac{\pi}{15}T \)

\( T = 7.5 \)