EE16B
Designing Information Devices and Systems II

Lecture 11A
Sampling and Interpolation
The Sampling Theorem
Interpolation with Basis Functions

\((x_i, y_i)\quad i = 1, 2, 3, \ldots\quad x_{i+1} - x_i = \Delta\quad \forall i\)

Define: \(\Phi(x)\)

\[\Phi(0) = 1\]

\[\Phi(k\Delta) = 0\quad k = \text{integer} \neq 0\]

\[y(x) = \sum_{k=-\infty}^{\infty} y_k \Phi(x - k\Delta)\]
Example:

• Linear Interpolation

\[ y(x) = \sum_{k=-\infty}^{\infty} y_k \Phi(x - k\Delta) \]
Example:

- Zero-Order Hold

\[ y(x) = \sum_{k=-\infty}^{\infty} y_k \Phi(x - k\Delta) \]
What is common to all these logos?
The sinc function

- Sinc:

\[
sinc(x) = \frac{\sin(\pi x)}{\pi x}
\]

\[
\Rightarrow \begin{cases} 
\frac{\sin(\pi x)}{\pi x} & x \neq 0 \\
1 & x = 0
\end{cases}
\]
The sinc function

• Let \( \Phi(x) = \text{sinc} \left( \frac{x}{\Delta} \right) \)

then \( \Phi(k\Delta) = \text{sinc}(k) = ? \)
The sinc function

- Let \( \Phi(x) = \text{sinc}\left(\frac{x}{\Delta}\right) \)

then \( \Phi(k\Delta) = \text{sinc}(k) = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases} \)

Interpolation with sinc: 
\[
y(x) = \sum_{k=-\infty}^{\infty} y_k \Phi(x - k\Delta)
\]
The sinc function

• Let \( \Phi(x) = \text{sinc} \left( \frac{x}{\Delta} \right) \)

then \( \Phi(k\Delta) = \text{sinc}(k) = \begin{cases} 
0 & k \neq 0 \\
1 & k = 0
\end{cases} \)

Interpolation with sinc:

\[
y(x) = \sum_{k=\infty}^{k=-\infty} y_k \Phi(x - k\Delta)
\]
Bandlimitedness

• The sinc function does not contain frequencies beyond a certain bandwidth

$$\text{sinc}(x) = \frac{1}{\pi} \int_0^\pi \cos(\omega x) d\omega$$

$$\left. \frac{\sin(\omega x)}{\pi x} \right|_0^\pi = \frac{\sin \pi x}{\pi x} \quad x \neq 0$$

Sinc is an infinite sum of cosine functions with frequencies in the range \( \omega \in [0, \pi] \)

More in EE120, EE123!
Sampling and Recovery

• Due to Shannon – Nyquist

CLAUSE SHANNON, THE FATHER OF THE INFORMATION AGE, TURNS 1100100

By Sibhan Roberts  April 20, 2016

Twelve years ago, Robert McEliece, a mathematician and engineer at Caltech, won the Claude E. Shannon Award, the highest honor in the field of information theory. During his acceptance lecture, at an international symposium in Chicago, he discussed the prize’s namesake, who died in 2001. Sometimes McEliece imagined, many rarely. Yet he still tinkered, in the time he might have spent cultivating the big reputation that scientists of his stature tend to seek. In 1973, the Institute of Electrical and Electronics Engineers christened the Shannon Award by bestowing it on the man himself, at the International Symposium on Information Theory in Ashkelon, Israel. Shannon had a bad case of nerves, but he pulled himself together and delivered a fine lecture on feedback, then dropped off the scene again. In 1985, at the International Symposium in

Sampling and Recovery

• Can we perfectly recover an analog signal from its samples?

Analog signal:
\[ y(x) = f(x) \]

Sample:
\[ y[n] = f(n\Delta) \]

Interpolate:
\[ \hat{f}(x) = \sum_{n=-\infty}^{\infty} y[n] \Phi(x - n\Delta) = ? f(x) \]
Sampling a sinusoid

• What rate should you be sampling a sinusoid?
Sampling Theorem

• If $f(x)$ is bandlimited by frequency $\omega_{\text{max}}$, then

$$f(x) = \hat{f}(x) = \sum_{n=-\infty}^{\infty} y[n] \Phi(x - n\Delta) \quad \Phi(x) = \text{sinc} \left( \frac{x}{\Delta} \right)$$

As long as,

$$\omega_{\text{max}} < \frac{\pi}{\Delta}$$

$$f_s = \frac{1}{\Delta} > 2 \frac{\omega_{\text{max}}}{2\pi} = 2f_{\text{max}}$$

$$\omega_s > 2\omega_{\text{max}}$$

Proof: EE120, EE123
Example 1

\[ f(x) = \cos\left(\frac{2\pi}{3} x\right) \quad \Delta = 1 \quad \omega_{\text{max}} < \frac{\pi}{\Delta} \]
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\[ f(x) = \cos\left(\frac{2\pi}{3} x\right) \quad \Delta = 1 \quad \omega_{\text{max}} < \frac{\pi}{\Delta} \]
Example 2

\[ f(x) = \cos\left(\frac{4\pi}{3} x\right) \quad \Delta = 1 \quad \omega_{\text{max}} < ? \frac{\pi}{\Delta} \]
Example 2

\[ f(x) = \cos\left(\frac{4\pi}{3}x\right) \quad \Delta = 1 \quad \omega_{\text{max}} < \frac{\pi}{\Delta} \]
Example 2

\[ f(x) = \cos\left(\frac{4\pi}{3} x\right) \quad \Delta = 1 \quad \omega_{\max} < \frac{\pi}{\Delta} \]

• Sinc interpolation gives:

\[ \hat{f}(x) = \cos\left(\frac{2\pi}{3} x\right) \]

Aliasing of high frequencies into lower ones!
Aliasing and Phase Reversal

\[ f(x) = \cos(\omega x + \phi) \quad \Delta = 1 \]

\[ y[n] = \cos(\omega n + \phi) \]

- Highest interpolated frequency will not be higher than \( \pi \)

\[ y[n] = \cos(\omega n + \phi) = \cos(2\pi n - (\omega n + \phi)) = \cos((2\pi - \omega)n - \phi) \]

\[ \cos(2\pi n - \theta) = \cos(\theta) \]

If \( \pi < \omega < 2\pi \) and \( \Delta = 1 \), there’s an equivalent lower frequency signal with the same samples!

\[ \hat{f}(x) = \cos((2\pi - \omega)x - \phi) \]
Example 2

\[ f(x) = \cos(\omega x + \phi) \quad \Delta = 1 \quad \omega = \frac{4\pi}{3} \quad \phi = 0 \]

\[ \hat{f}(x) = \cos((2\pi - \omega)x - \phi) \]

\[ = \cos\left(\frac{2\pi}{3}x\right) \]
Example 3

\[ f(x) = \sin(1.9\pi x) \quad \Delta = 1 \]

\[ = \cos(1.9\pi x - \frac{\pi}{2}) \]

\[ \hat{f}(x) = \cos(0.1\pi x + \frac{\pi}{2}) \]

\[ = -\sin(0.1\pi x) \]