1. RLC circuit

In this question, we will take a look at an electrical systems described by second order differential equations and analyze it using the phasor domain. Consider the circuit below where \( V_s \) is a sinusoidal signal, \( L = 1\, \text{mH} \) and \( C = 1\, \text{nF} \):

(a) Transform the circuit into phasor domain.

**Solution:**

\[
\begin{align*}
Z_R &= R \\
Z_L &= j\omega L \ \\
Z_C &= \frac{1}{j\omega C} \\
\end{align*}
\]

(b) Solve for the transfer function \( H_C(\omega) = \frac{V_C}{V_s} \) in terms of \( R, L, \) and \( C \)

**Solution:** \( V_C \) is a voltage divider where the output voltage is taken across the capacitor.

\[
\begin{align*}
\bar{V}_C &= \frac{Z_C}{Z_R + Z_L + Z_C} \bar{V}_s \\
H_C(\omega) &= \frac{Z_C}{Z_R + Z_L + Z_C} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}} \\
\end{align*}
\]

Multiplying by numerator and denominator by \( j\omega C \)

\[
H_C(\omega) = \frac{1}{(j\omega)^2LC + j\omega RC + 1}
\]
(c) Solve for the transfer function $H_L(\omega) = \frac{\tilde{V}_L}{\tilde{V}_s}$ in terms of $R$, $L$, and $C$

**Solution:** $\tilde{V}_L$ is a voltage divider where the output voltage is taken across the inductor.

$$\tilde{V}_L = \frac{Z_L}{Z_R + Z_L + Z_C} \tilde{V}_s$$

$$H_L(\omega) = \frac{Z_L}{Z_R + Z_L + Z_C} = \frac{j\omega L}{R + j\omega L + \frac{1}{j\omega C}}$$

Multiplying by numerator and denominator by $j\omega C$

$$H_L(\omega) = \frac{(j\omega)^2 LC}{(j\omega)^2 LC + j\omega RC + 1}$$

(d) Solve for the transfer function $H_R(\omega) = \frac{\tilde{V}_R}{\tilde{V}_s}$ in terms of $R$, $L$, and $C$

**Solution:** $\tilde{V}_R$ is a voltage divider where the output voltage is taken across the resistor.

$$\tilde{V}_R = \frac{Z_R}{Z_R + Z_L + Z_C} \tilde{V}_s$$

$$H_R(\omega) = \frac{Z_R}{Z_R + Z_L + Z_C} = \frac{R}{R + j\omega L + \frac{1}{j\omega C}}$$

Multiplying by numerator and denominator by $j\omega C$

$$H_R(\omega) = \frac{j\omega RC}{(j\omega)^2 LC + j\omega RC + 1}$$

(e) Sketch the Bode plots of $H_C(\omega)$, $H_L(\omega)$, and $H_R(\omega)$ when $R = 2k\Omega$

**Solution:**

$H_C(\omega)$:

Let’s look at the poles. It is a second order transfer function in the form of

$$\left( \frac{j\omega}{\omega_c} \right)^2 + j\omega \frac{2\xi}{\omega_c} + 1$$

Where

$$\omega_c = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} 10^{-9}}} = 10^6 \text{ rad/s}$$

which means there are 2 poles at $10^6 \text{ rad/s}$

Zeros:

Since the numerator has no frequency terms, there are no zeros

We have 2 poles and no zeros, which means this will be a low pass filter with a cutoff freq of $10^6 \text{ rad/s}$, with a 40 dB/dec rolloff.
To find the magnitude in the flatband for a low pass filter, we look at the magnitude when all frequency terms are 0. For this filter, we just get a magnitude of $\frac{1}{1} = 1 = 0$ dB.

As for phase, the two poles will cause a phase shift starting at $10^5$ rad/s at a rate of $-90^\circ$/dec ($-45^\circ$/dec per pole) until a decade past the corner frequency. This results in the Bode plot below.

![Bode plot of $H_C(\omega)$](image)

**Figure 1: Bode plot of $H_C(\omega)$**

$H_L(\omega)$:
- **Poles**: same as $H_C(\omega)$, 2 poles at $10^6$ rad/s
- **Zeros**: Since there’s only a frequency term, the zero is located at 0 rad/s. Since $\omega$ is squared, there are 2 zeros located at 0 rad/s. This causes an initial phase shift of $180^\circ$ ($90^\circ$ per zero at 0 rad/s)

This results in a high pass filter with a corner frequency at $10^6$ rad/s, where the roll off is 40 dB/dec.

To find the magnitude in the flatband for a high pass filter, we look at the magnitude when everything but the highest order frequency terms are zero. Since this transfer function is 2nd order, we look at the $\omega^2$ terms and get a magnitude of $\frac{(j\omega)^2LC}{(j\omega)^2LC} = 1 = 0$ dB.
$H_L(\omega)$:
Poles: same as others, 2 poles at $10^6$ rad/s
Zeros: Like $H_L(\omega)$, it has a zero at 0 rad/s, but this time there’s only 1.
This creates a bandpass filter. To find the peak magnitude, we look at the first asymptote. Before $\omega_c$, the only pole or zero that affects $|H(\omega)|$ is the zero at 0 rad/s:

$$|H_R(\omega \leq \omega_c)| = |j\omega RC|$$

Plugging in $\omega = \omega_c$ will give us the peak value of the transfer function:

$$|H_R(\omega = \omega_c)| = |j\omega_c RC| = \left| \frac{jRC}{\sqrt{LC}} \right| = \left| \frac{jR\sqrt{C}}{\sqrt{L}} \right|$$

$$|H_R(\omega = \omega_c)| = \frac{2000\Omega \sqrt{10^{-9}F}}{\sqrt{10^{-3}H}} = 2 = 20\log_{10}(2) \text{ dB} = 6.02 \text{ dB}$$
(f) Draw the Bode plot of $H_R(\omega)$ two more times, but change $R$ to be 20Ω, then 200kΩ.

**Solution:** Looking back at our solution in part (e), the only difference is the peak value of $H(\omega)$.

Plugging the new resistor values in, we get:

$$R = 20\Omega \rightarrow |H(\omega_c)| = \frac{20\sqrt{10^{-9}}}{\sqrt{10^{-3}}} = -33.98\text{dB}$$

$$R = 200k\Omega \rightarrow |H(\omega_c)| = \frac{2 \times 10^5 \sqrt{10^{-9}}}{\sqrt{10^{-3}}} = 46.02\text{dB}$$

Every other part of the bode plot is the same as the plot in part (e)
(g) Find $\omega_{c1}$ and $\omega_{c2}$ of $H_R(\omega)$ for $R = 20\Omega$, $R = 2k\Omega$ and $R = 200k\Omega$

**Solution:** While we said the corner frequencies were at $10^6$ rad/s in part (e), that was only for the Bode plot approximation. $10^6$ rad/s is actually the center frequency. To get the corner frequencies, we
need to find the values of \( \omega \) where \( |H_R(\omega)| = \frac{1}{\sqrt{2}} \)

\[
\omega_{c1,2} = \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \pm \frac{R}{2L}
\]

\( R = 20\Omega \rightarrow \omega_{c1} = 9.9005 \times 10^5 \text{ rad/s}, \omega_{c2} = 1.01 \times 10^6 \text{ rad/s} \)
\( R = 2000\Omega \rightarrow \omega_{c1} = 4.1421 \times 10^5 \text{ rad/s}, \omega_{c2} = 2.4142 \times 10^6 \text{ rad/s} \)
\( R = 200000\Omega \rightarrow \omega_{c1} = 0.05 \times 10^5 \text{ rad/s}, \omega_{c2} = 2 \times 10^8 \text{ rad/s} \)

(h) Which of the three values of \( R \) gives the largest bandwidth? Which gives the highest \( Q \)?

**Solution:**

\[
BW = \omega_{c2} - \omega_{c1}
\]
\( BW(20\Omega) = 2 \times 10^4 \text{ rad/s} \)
\( BW(2000\Omega) = 2 \times 10^6 \text{ rad/s} \)
\( BW(200000\Omega) = 2 \times 10^8 \text{ rad/s} \)

\( R = 200 \text{ k}\Omega \) results in the highest bandwidth

\[
Q = \frac{\omega_0}{BW} = \frac{10^6 \text{ rad/s}}{BW}
\]
\( Q(20\Omega) = 50 \)
\( Q(2000\Omega) = 0.5 \)
\( Q(200000\Omega) = 0.005 \)

\( R = 20\Omega \) results in the highest \( Q \).

(i) Graph the actual frequency response of \( H_R(\omega) \) for \( R = 20\Omega, R = 2\text{k}\Omega \) and \( R = 200\text{k}\Omega \). What is different between the Bode plots and the actual responses? Label on each graph what type of damping the circuit experiences.

**Solution:**
Figure 6: $R = 20\Omega$ frequency response

Figure 7: $R = 2\, k\Omega$ frequency response
The three frequency responses are now different from each other. The peak gain for each case is now 0dB, instead of varying. This is because the Bode approximation ignores the $\omega$ term in the quadratic pole equation. However, the different values of $\xi$ affect the gain in the passband significantly. The passband of each one corresponds to the bandwidth of the filter. The phase no longer is a constant $-90^\circ$/dec drop, it now depends on the resistor value. The frequency responses are different because the Bode plots only look at the asymptotes of the frequency response; it doesn’t take into account things like Q or bandwidth. For second order and higher transfer functions, Bode plots are not always the most accurate.

To determine the damping, we can use the same equations in the past:

$$\alpha = \frac{R}{2L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

we know the system is critically damped when

$$\alpha = \omega_0$$

Using substitution, we get

$$R = 2\sqrt{\frac{L}{C}} = 2\, \text{k}\Omega$$
since $R$ is directly proportional to $\alpha$, we know if $R < 2 \, k\Omega$, then the system is underdamped, while if $R > 2 \, k\Omega$, then the system is overdamped.

$$ R = 20 \Omega \rightarrow \text{underdamped} $$

$$ R = 2 \, k\Omega \rightarrow \text{critically damped} $$

$$ R = 200 \, k\Omega \rightarrow \text{overdamped} $$

Another way to think about damping is to look at the zeros of the quadratic. Back with the differential equations, we saw that the system was underdamped if the solution to the 2nd order equation was complex, it was critically damped if the solutions were the same, and over damped if the solutions were purely real. These conditions still hold for the quadratic pole.

$$ \omega = \frac{-2\xi \omega_0 \pm \sqrt{(2\xi \omega_0)^2 - 4 \omega_0^2}}{2\omega_0^2} $$

The determining factor on whether $\omega$ is complex is the square root term.

$$ \sqrt{4\xi^2 \omega_0^2 - 4 \omega_0^2} = \sqrt{\frac{4}{\omega_0^2} (\xi^2 - 1)} $$

If $\xi^2 \leq 1$, then $\omega$ will be complex, which means the system is underdamped. If $\xi^2 = 1$, then the square root term will be zero, which means $\omega$ will have repeated roots. This corresponds to a critically damped system. Finally, if $\xi^2 \geq 1$, then the square root term is real, and the roots of $\omega$ are both real, which is an overdamped system. By taking each condition and taking the square root of each side, we can say:

$$ |\xi| \leq 1 \rightarrow \text{Underdamped} $$

$$ |\xi| = 1 \rightarrow \text{Critically damped} $$

$$ |\xi| \geq 1 \rightarrow \text{Overdamped} $$

Looking back at our transfer function $H_R(\omega)$, we see

$$ \xi = \frac{R \sqrt{C}}{2\sqrt{L}} $$

Plugging into the conditions and isolating $R$, we get

$$ |R| \leq \left| \frac{2\sqrt{L}}{\sqrt{C}} \right| \rightarrow \text{Underdamped} $$

$$ |R| = \leq \left| \frac{2\sqrt{L}}{\sqrt{C}} \right| \rightarrow \text{Critically damped} $$

$$ |R| \geq \left| \frac{2\sqrt{L}}{\sqrt{C}} \right| \rightarrow \text{Over damped} $$
2. Circuit Design

In this problem, you will find a circuit where several components have been left blank for you to fill in. Assume the op-amp is ideal.

You have at your disposal only one of each of the following components (not including \( R_1 \) and \( R_2 \)):

- (a) an open circuit
- (b) a short circuit
- (c) a resistor (you choose from the value \( R = 1\,\text{k}\Omega, 15\,\text{k}\Omega, 30\,\text{k}\Omega \))
- (d) a capacitor (you choose from the value \( C = 0.5\,\text{\mu F}, 1\,\text{\mu F}, 2\,\text{\mu F} \))

Consider the circuit below. The voltage source \( v_{in}(t) \) has the form \( v_{in}(t) = v_0 \cos(\omega t + \phi) \). The labeled voltages \( V_{in}(\omega) \) and \( V_{out}(\omega) \) are the phasor representation of \( v_{in}(t) \) and \( v_{out}(t) \). The transfer function \( H(\omega) \) is defined as \( H(\omega) = V_{out}(\omega)/V_{in}(\omega) \).

(a) Let \( R_1 \) be \( 1\,\text{k}\Omega \). **Fill in the boxes and determine the value of \( R_2 \) so that**

- It is a high-pass filter.
- \(|H(\infty)| = 10\).
- \(|H(10^3)| = \sqrt{50}\).
- \( R_2 \) must be one of the three values listed above

**Solution:**

Let the left box be \( Z_1 \) and the right box be \( Z_2 \). The circuit should be a high-pass filter, so \( Z_2 \) cannot be a short circuit or a capacitor (otherwise \( V_{out}(\infty) = 0 \)). Since \( Z_2 \) cannot be a capacitor, \( Z_1 \) must be a capacitor otherwise we would not have a zero or pole in our circuit, which means it wouldn’t be a filter. \( Z_2 \) is either an open circuit or a resistor. Let \( R_f = R_2 \parallel Z_2 \) and \( Z_1 = \frac{1}{j\omega C} \). The transfer function is given by

\[
H(\omega) = -\frac{R_f}{R_1 + \frac{1}{j\omega C}} = -\frac{j\omega R_f C}{1 + j\omega R_1 C} = \frac{-R_f}{R_1} \frac{j\omega R_1 C}{1 + j\omega R_1 C}
\]
Observing the transfer function, we know $H(0) = 0$ and $H(\infty) = -\frac{R_f}{R_1}$ so it is a high-pass filter. From $|H(\infty)| = 10$, we know $R_f = 10R_1 = 10 \, \text{k}\Omega = R_2 \parallel Z_2$.

Since $10 \, \text{k}\Omega$ is not an option for the resistor values, we know that $Z_2$ must be a resistor in order to get the correct gain. The equivalent resistance of two parallel resistors can never go higher than the smallest resistor in the parallel combination. This means $R_2$ and $Z_2$ cannot be $1 \, \text{k}\Omega$. That leaves us with three options, both are $15 \, \text{k}\Omega$, both are $30 \, \text{k}\Omega$, or one is $30 \, \text{k}\Omega$ and the other is $15 \, \text{k}\Omega$. If you have two resistors with the same value in parallel, the equivalent resistance is half the original value. This means $R_2$ and $Z_2$ cannot be $1 \, \text{k}\Omega$. That leaves us with three options, both are $15 \, \text{k}\Omega$, both are $30 \, \text{k}\Omega$, or one is $30 \, \text{k}\Omega$ and the other is $15 \, \text{k}\Omega$. If you have two resistors with the same value in parallel, the equivalent resistance is half the original value. So if both were $15 \, \text{k}\Omega$, the equivalent resistance would be $7.5 \, \text{k}\Omega$, which is too low. For the case where they’re both $30 \, \text{k}\Omega$, you get $15 \, \text{k}\Omega$, which is too high. This means the resistor pairing must be a $15 \, \text{k}\Omega$ and $30 \, \text{k}\Omega$ resistor. We can also double check using the parallel resistor equation:

$$R_1 \parallel R_2 = (R_1^{-1} + R_2^{-1})^{-1}$$

$$15000 \, \text{k}\Omega \parallel 30000 \, \text{k}\Omega = \frac{1}{\frac{1}{15000 \, \text{k}\Omega} + \frac{1}{30000 \, \text{k}\Omega}} = \frac{1}{\frac{1}{30000 \, \text{k}\Omega}} = 10000 \, \text{k}\Omega$$

To figure out the capacitance value for $Z_1$, we look at the boundary condition for $|H(10^3)| = \sqrt{50}$. Let $x = 10^3R_1C$

$$\sqrt{50} = 10 \frac{\sqrt{x^2}}{\sqrt{1+x^2}} \Rightarrow \frac{1}{2} = \frac{x^2}{1+x^2} \Rightarrow x^2 = 1 \Rightarrow 10^3R_1C = 1 \Rightarrow C = \frac{1}{10^3R_1} = 1 \mu\text{F}$$

Thus,

- $R_2 = 15 \, \text{k}\Omega$ (or $30 \, \text{k}\Omega$).
- The right box: a $30 \, \text{k}\Omega$ resistor (or $15 \, \text{k}\Omega$ resistor if you said $R_2$ was $30 \, \text{k}\Omega$).
- The left box: a capacitor with $C = 1 \mu\text{F}$.

(b) Draw the Bode plot of this transfer function.

**Solution:** Using the values found in part (a), we end up with the transfer function

$$H(\omega) = -\frac{R_f}{R_1} \frac{j\omega R_1 C}{1 + j\omega R_1 C} = -10 \frac{j\omega}{1 + j\omega 10^3}$$

This means we have 1 zero at $0 \, \text{rad/s}$ and 1 pole at $10^3 \, \text{rad/s}$. We know at high frequencies, the magnitude is 10, which is $20 \, \text{dB}$. We have a negative sign in the DC gain, which contributes $-180^\circ$ to the phase at $0 \, \text{rad/s}$. We also have a zero at $0 \, \text{rad/s}$ which contributes $90^\circ$ at $0 \, \text{rad/s}$, which means our initial phase is $-90^\circ$.
3. Bode plots to Transfer Functions

(a) Find the transfer function $H(\omega)$ that corresponds to the magnitude plot below. The phase of $H(\omega)$ is $90^\circ$ at $\omega = 0$.

Solution: The first step to figuring out the transfer function is to determine what frequency your poles and zeros are at, and how many you have.
Looking at the graph starting at low frequencies, we see the first corner frequency is at $10^3$ rad/s. Before that point, the transfer function is rising at a rate of 60 dB/dec. Since $H(\omega)$ is increasing before the first corner frequency, we know we have zeros at 0 rad/s. Each pole gives a slope of 20 dB/dec, so we know there has to be 3 zeros at 0 rad/s to get the 60 dB/dec rise.

At $10^3$ rad/s, the slope flattens to 0 dB/dec, which means there must be 3 poles to counteract the zeros. The next corner frequency is at $10^5$ rad/s. From here, $H(\omega)$ starts falling off at 20 dB/dec, which means there’s one pole.

The last corner frequency is at $10^6$ rad/s, where $H(\omega)$ starts dropping at -80 dB/dec. We already had a slope of -20 dB/dec, so at $10^6$ rad/s, we gained an additional -60 dB/dec slope. This means there are 3 more poles at $10^6$ rad/s.

For poles and zeros, in total we have:
- 3 zeros at 0 rad/s
- 3 poles at $10^3$ rad/s
- 1 pole at $10^5$ rad/s
- 3 poles at $10^6$ rad/s

These poles and zeros give us a transfer function of

$$H(\omega) = \frac{(j\omega)^3}{(1 + \frac{j\omega}{10^3})^3(1 + \frac{j\omega}{10^5})(1 + \frac{j\omega}{10^6})^3}$$

The next thing we need to consider is the gain of the transfer function. To determine what DC gain we need to multiply our current transfer function by to get the correct passband gain, we need to take our current $H(\omega)$, and plug in an $\omega$ value in one of the regions to determine the gain. If we look at the far left region, where $\omega \leq 10^3$ rad/s, The dominating poles and/or zeros in that region are the poles and zeros at 0 rad/s, so the magnitude of our transfer function can be approximated to

$$|H(\omega \leq 10^3)| = A |(j\omega)^3|$$

where A is some DC gain. If we plug in $\omega = 10^3$, we should get $|H(10^3)| = 12$ dB.

$$|H(10^3)| = 12\text{dB} = A \cdot 10^9$$

$$A = \frac{12\text{dB}}{10^9} = \frac{10^{12}}{10^9} = 3.98 \cdot 10^{-9}$$

The DC gain of the transfer function is $3.98 \cdot 10^{-9}$.

The last thing we need to consider is the initial phase. The problem states that the phase of $H(0) = 90^\circ$. Our transfer function has 3 zeros at 0 rad/s. Each zero adds $90^\circ$ phase shift at $\omega = 0$, which would mean we currently have a phase of $270^\circ$, which is equivalent to $-90^\circ$ (since phase starts wrapping around at $180^\circ$). To get to the correct phase at 0 rad/s, we need to shift the response $180^\circ$, which is equivalent to changing our DC gain to a negative gain.

This leaves us with the final transfer function

$$H(\omega) = -3.98 \cdot 10^{-9} \frac{(j\omega)^3}{(1 + \frac{j\omega}{10^3})^3(1 + \frac{j\omega}{10^5})(1 + \frac{j\omega}{10^6})^3}$$
(b) Find the transfer function $H(\omega)$ that corresponds to the magnitude plot below. The phase of $H(\omega)$ is $0^\circ$ at $\omega = 0$

![Magnitude graph for 3 (b)](image)

**Solution:** The first corner frequency is at $10^2$ rad/s. Before that, $H(\omega)$ is dropping at -40 dB/dec, which means there are 2 poles at 0 rad/s.

At $10^2$ rad/s, it flattens out, so there must be 2 zeros at $10^2$ rad/s to cancel out the 2 poles.

At $10^3$ rad/s, $H(\omega)$ starts rising at 20 dB/dec, which means there’s 1 zero at $10^3$ rad/s.

At $10^4$ rad/s, it flattens out again, so there must be 1 pole at $10^4$ rad/s to cancel out the zero at $10^3$ rad/s.

At $10^5$ rad/s, it starts decreasing again at -20 dB/dec. This means there is 1 pole at $10^5$ rad/s.

At $10^6$ rad/s, the slope flattens out again, which means there’s a zero at $10^6$ rad/s to cancel out the pole at $10^5$ rad/s. At $10^7$ rad/s, the slope increases to 40 dB/dec, which means there must be 2 zeros at $10^6$ rad/s.

For poles and zeros, in total we have:

- 2 poles at 0 rad/s
- 2 zeros at $10^2$ rad/s
- 1 zero at $10^3$ rad/s
- 1 pole at $10^4$ rad/s
- 1 pole at $10^5$ rad/s
- 1 zero at $10^6$ rad/s
- 2 zeros at $10^7$ rad/s

This gives us a transfer function of

$$H(\omega) = \frac{(1 + \frac{j\omega}{10^2})(1 + \frac{j\omega}{10^3})(1 + \frac{j\omega}{10^4})(1 + \frac{j\omega}{10^5})}{(j\omega)^2(1 + \frac{j\omega}{10^3})(1 + \frac{j\omega}{10^4})}$$

Next, we find the DC gain. This can be done by looking at $|H(\omega)|$ for $\omega \leq 10^2$ rad/s, which is dominated by the poles at 0 rad/s

$$|H(\omega \leq 10^2)| = A \left| \frac{1}{(j\omega)^2} \right|$$
plugging in 10^2 rad/s, we should get -100 dB

\[ H(10^2) = -100\text{dB} = A \frac{1}{(10^2)^2} \]

\[ A = -100\text{dB} \times 10^4 = 10^{-100} \times 10^4 = 10^{-1} \]

Now we need to correct for the phase at 0 rad/s. From the 2 poles at 0 rad/s, we have an initial phase shift of \(-180^\circ\). We need a phase shift of 0 at 0 rad/s, so we need an additional 180^\circ shift. This can be done by multiplying our DC gain by -1.

The final transfer function is

\[ H(\omega) = -10^{-1}(1 + \frac{j\omega}{500})^2(1 + \frac{j\omega}{10^3})(1 + \frac{j\omega}{10^3})^2 \]

\[ (j\omega)^2(1 + \frac{j\omega}{10^2})(1 + \frac{j\omega}{10^2}) \]

(c) Using Python or MATLAB, graph the bode plot for the transfer function

\[ H(\omega) = \frac{(500 + j\omega)(10^3 + j\omega)^2}{(j\omega)(10^2 + j\omega)^3} \]

Solution: First thing we need to do is to transform \( H(\omega) \) into standard Bode plot form:

\[ H(\omega) = \frac{500(1 + \frac{j\omega}{500})(10^3)^2(1 + \frac{j\omega}{10^2})^2}{(j\omega)(10^2)^3(1 + \frac{j\omega}{10^2})^3} = 500(1 + \frac{j\omega}{500})(1 + \frac{j\omega}{10^2})^2 \]

There’s 1 pole at 0 rad/s, so \( H(\omega) \) starts dropping off by -20 dB/dec until it hits its first corner frequency at 10^2 rad/s. Three more poles add to the slope, and \( H(\omega) \) starts dropping off by -80 dB/dec until it hits the next corner frequency. At 500 rad/s, there is a zero, which counteracts one of the poles, so \( H(\omega) \) will start dropping off by -60 dB/dec until the next corner frequency. At 10^3 rad/s, 2 more zeros appear and decrease the slope to -20 dB/dec. This was the last corner frequency, so \( H(\omega) \) will continue to drop off by -20 dB/dec.

Now we know all the slopes and relative values of the magnitudes across frequency, but we need to plug in a value of \( \omega \) to get an absolute value. Using \( H(\omega) \) for \( \omega \leq 10^2 \) rad/s, we can say

\[ |H(10^2)| = 20\log_{10}(\frac{500}{j10^2}) = 13.979\text{dB} \]

Now we have everything we need to graph the Bode plot.
If you rather plot the actual frequency response, you get the following plot:
4. LTE Filter

We are interested in creating a filter to receive LTE signals (a.k.a 4G signals often used for phones). There are many different frequencies at which signals are transmitted (called bands), so we’re going to focus on LTE band 1. Band 1 transmits with a center frequency of 2.140 GHz, and a bandwidth of 60 MHz.

(a) Find $\omega_1$ and $\omega_2$ for LTE band 1.

**Solution:**

$$f_{c1,2} = f_0 \pm \frac{BW}{2} = 2.140 \times 10^9 \pm 30 \times 10^6 = 2.110 \times 10^9, 2.170 \times 10^9 \text{ Hz}$$

$$w_{c1,2} = 2\pi f_{c1,2} = 13.258 \text{ Grad/s, 13.635 Grad/s}$$

(b) In order to receive a good signal, we need to ensure that $|H(\omega)|$ is small enough outside the pass band. Otherwise, noise from neighboring transmission bands will distort the signal we’re trying to receive. Let’s say we did some analysis, and found that at 200MHz away from the center frequency (1.940 GHz and 2.340 GHz), our $|H(\omega)|$ has to be 80dB lower than the pass band gain. Using your frequency response and Bode plot knowledge, determine the minimum number of poles and zeros required to get the required magnitude drop. **Solution:** For simplicity, we assume that the filter is symmetrical, so we only have to look at one half the filter, then mirror it to get the other half. We will look at the lower half for this solution.

The first thing we need to do is to get the frequency into a radial frequency:

$$\omega_d = 2\pi f = 2\pi \times 1.940 \times 10^9 = 1.2189 \times 10^{10} \text{ rad/s}$$

For a Bode approximation, $|H(\omega)|$ at the corner frequencies are at the passband gain, so we know we have an 80 dB drop over a change in frequency. However, we need to convert to the frequencies to logarithmic scale to get the slope in terms of dB/dec.

$$\text{slope} = \frac{80 \text{ dB}}{\log_{10}(\omega_1) - \log_{10}(\omega_d)} = 2193 \text{ dB/dec}$$

Each zero gives 20 dB/dec, which means we would need 110 zeros to achieve this roll off! In reality, this is not true, since we did not account for Q in our Bode estimations. Continuing with the 110 number, we need 110 zeros for the roll off, 110 poles at $\omega_1$ to get the flat band, and 110 more poles at $\omega_2$ to get the roll off on the upper half of the filter, which is a total of 330 poles and zeros.

It turns out that our assumption for the circuit being symmetrical is not a very good assumption. Now, let’s look how many poles we’d need on the higher half of the filter to get the 80dB rejection. All the equations are the same, but with the upper values instead of lower:

$$\text{slope} = \frac{80 \text{ dB}}{\log_{10}(\omega_2) - \log_{10}(\omega_d)} = \frac{80 \text{ dB}}{\log_{10}(2.17 \times 10^9 \times 2\pi) - \log_{10}(2.34 \times 10^9 \times 2\pi)} = -2442 \text{ dB/dec}$$

It turns out we need 123 poles on the higher end of the filter to achieve the rejection required, which would give us a total of 343 poles and zeros.
(c) Assuming we have a pass band gain of 1, give an expression for \( H(\omega) \)

**Solution:**

If you used the symmetrical filter transfer function:

\[
H(\omega) = \frac{(j\omega)^{110}}{(1 + j\omega\omega_1)^{110}(1 + j\omega\omega_2)^{110}} = \frac{(j\omega)^{110}}{(1 + \frac{j\omega}{13.258 \times 10^9})^{110}(1 + \frac{j\omega}{13.635 \times 10^9})^{110}}
\]

If you used the full non-symmetrical filter:

\[
H(\omega) = \frac{(j\omega)^{110}}{(1 + j\omega\omega_1)^{110}(1 + j\omega\omega_2)^{123}} = \frac{(j\omega)^{110}}{(1 + \frac{j\omega}{13.258 \times 10^9})^{110}(1 + \frac{j\omega}{13.635 \times 10^9})^{123}}
\]

5. Color organ filter design

Consider another microphone similar to the one which we were analyzing last week. As before, you obtain the data by playing a uniform tone with varying frequencies, and measuring the resultant peak-to-peak voltages using an oscilloscope. Below is the data obtained from your experiments:

<table>
<thead>
<tr>
<th>Input frequency (Hz)</th>
<th>Output peak-to-peak (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.4</td>
</tr>
<tr>
<td>20</td>
<td>1.5</td>
</tr>
<tr>
<td>40</td>
<td>1.5</td>
</tr>
<tr>
<td>60</td>
<td>1.6</td>
</tr>
<tr>
<td>100</td>
<td>2.2</td>
</tr>
<tr>
<td>160</td>
<td>2.3</td>
</tr>
<tr>
<td>320</td>
<td>2.4</td>
</tr>
<tr>
<td>640</td>
<td>2.5</td>
</tr>
<tr>
<td>1200</td>
<td>5</td>
</tr>
<tr>
<td>2500</td>
<td>5</td>
</tr>
<tr>
<td>5000</td>
<td>5</td>
</tr>
<tr>
<td>10000</td>
<td>4.9</td>
</tr>
</tbody>
</table>

(a) Now that you know how to design filters and basic op-amp amplifiers, please design the filters and amplifiers (coloured yellow and red respectively) for the color organ circuit below. You may use one op-amp per coloured block. Please draw the schematic-level representation (with op-amps and resistors/capacitors) of your designs for the filters and amplifiers, and show your work for the any values for \( R_s \) and \( C_s \). If possible, choose ‘realistic’ values for resistors (1kOhm to 500kOhm) and capacitors (1nF to 200uF) or justify your reasoning.

Don’t forget about the filter between the two parts of the band-pass filter! (Why is it necessary?)

For the purposes of this problem, consider the following frequency ranges for each filter:

- Low pass filter - less than 100 Hz
- High pass filter - more than 2400 Hz
- Band pass filter - between 300 and 600 Hz
In the lab, you will use different frequencies for your cutoff as each microphone will be different. For the purposes of this homework problem, to ensure that the LED turns on properly, the output (as measured before the 10Ω resistors) in the desired ranges should have a peak-to-peak of about 5V, and any output outside the desired ranges caused by identical-amplitude signals from the other ranges should be less than 1.5V peak-to-peak.

**Solution:** First, we'll look at the amplifying blocks of each branch. Each branch needs a different gain to reach 5V. The lowpass filter needs a gain of \( \frac{5V}{1.5V} = 3.33 \), the high pass needs \( \frac{5}{5} = 1 \), and the bandpass needs \( \frac{5}{2.5} = 2 \) (if you found required gain using 2.4 instead of 2.5, that’s fine, I’m rounding for simplicity). Since the highpass branch doesn’t need a gain, we can use a buffer stage for its amplifier block.

![Buffer circuit](image)

**Figure 16: Buffer circuit**

As for the low pass and bandpass branches, we’ll use a non-inverting op amp for the gain stage. We use a non-inverting op-amp since the input resistance of the circuit in theoretically infinite, so it shouldn’t affect the stage before it.

---

1Note that LEDs will only turn on when the voltage across it is greater than say **positive** 0.7V. If you wanted to detect and turn on the LED all the time instead of half the time, you would use either a mini rectifier, some kind of peak detection circuit, or digital processing, all of which is beyond the scope of the course.
The gain of a non-inverting amplifier is

\[ A_v = 1 + \frac{R_f}{R_s} \]

Setting \( R_s = 1 \, \text{k}\Omega \), we see that the lowpass needs \( R_f = 2.33 \, \text{k}\Omega \) and the bandpass needs \( R_f = 1 \, \text{k}\Omega \).

Next, let’s look at the low pass filter. The low pass filter needs a cutoff frequency of 100 Hz. Using a resistor and capacitor, we can make a low pass filter.

The transfer function of this filter is just a voltage divider:

\[
H(\omega) = \frac{Z_C}{Z_C + Z_R} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + \frac{1}{R}} = \frac{1}{1 + j\omega RC}
\]

The transfer function has a pole at \( \omega_c = \frac{1}{RC} \) rad/s. So we need to choose an \( R \) and \( C \) such that

\[
\frac{1}{RC} = 100 \ast 2\pi \text{ rad/s}
\]

If we choose a value of \( R = 1 \, \text{k}\Omega \), we get \( C = 1.6 \mu\text{F} \).

Next, let’s look at the high pass filter. We can make a high pass filter by taking the same circuit and taking the output voltage across the resistor instead of the capacitor.
The transfer function for this circuit is

\[
\frac{j\omega RC}{1 + j\omega RC}
\]

which has the corner frequency of \( \omega_c = \frac{1}{RC} \), which is the same equation as the low pass filter. To get a corner frequency of 2.4 kHz, we can set \( R = 1 \, \text{k}\Omega \), which will give us \( C = 66.3 \, \text{nF} \).

Finally, let's look at the bandpass filter. We can create a bandpass by feeding a low pass filter’s output into a high pass filter’s input. Since a high pass filter rejects all frequencies lower than its corner frequency, and a low pass rejects all frequencies higher than its corner frequency, the cascaded circuit will only pass signals in between the high pass’ center frequency and the low pass’ frequency. This means if we set the high pass’ corner frequency to be 300 Hz and the low pass’ corner frequency to be 600 Hz, the resulting filter will pass only signals between those two frequencies, which is what we desire. While feeding a low pass into a high pass is what we want, we still need one more thing—a buffer stage between the two filters. We need this buffer because if you were to directly connect the filters together, their impedances would directly impact each other, creating a totally new transfer function. Over all, the filter would look something like this:

![Bandpass Filter Diagram](image)

\[ V_{in} \quad R_L \quad C_L \quad + \quad - \quad C_H \quad - \quad R_H \quad V_{out} \]

\( R_L \) and \( C_L \) are the low pass resistor and capacitor while \( R_H \) and \( C_H \) are the high pass resistor and capacitor. As we saw for the previous blocks, the corner frequency of both the high pass and the low pass is \( \frac{1}{\text{RC}} \). So we can say

\[
\frac{1}{R_L C_L} = 600 \times 2\pi
\]

\[
\frac{1}{R_H C_H} = 300 \times 2\pi
\]
if we set both \( R_L \) and \( R_H \) to 1 kΩ, we can solve for the capacitances and get \( C_L = 265.3 \) nF and \( C_H = 530.5 \) nF

(b) For each of the waveforms from the microphone, determine the output voltage for each filter right before the 10Ω resistors (\( V_{out} \)).

\( V_i(t) = 0.75 \cos(2\pi \cdot 50Hz \cdot t) \)
\( V_b(t) = 1.25 \cos(2\pi \cdot 450Hz \cdot t) \)
\( V_h(t) = 2.5 \cos(2\pi \cdot 3000Hz \cdot t) \)

**Solution:** To get the output voltage of each branch, we find the total transfer function for each branch and multiply it by the input signal. The total transfer function of a branch is the transfer function of the filter multiplied by the gain of the amp. stage. We will denote low pass values with L subscripts, band pass with B subscripts, and high pass with H subscripts.

\[
H_L(\omega) = A_v \frac{1}{1 + j\omega RC} = 3.33 \frac{1}{1 + \frac{j\omega}{100\pi}}
\]
\[
H_H(\omega) = A_v \frac{j\omega RC}{1 + j\omega RC} = 1 - \frac{j\omega}{2400 + j\omega}
\]
\[
H_B(\omega) = A_v \frac{j\omega R_HC_H}{(1 + j\omega R_LC_L)(1 + j\omega R_HC_H)} = 2 \frac{j\omega}{(300 + j\omega)(300 + j\omega)}
\]

We only care about the magnitude of the output voltages, so we can say

\[
|V_{o,L}| = |V_{in}(\omega)||H_L(\omega)|
\]
\[
|V_{o,H}| = |V_{in}(\omega)||H_H(\omega)|
\]
\[
|V_{o,B}| = |V_{in}(\omega)||H_B(\omega)|
\]

For \( V_i(t) \):

\[
|V_{o,L}| = |V_i(2\pi \cdot 50)||H_L(2\pi \cdot 50)| = 0.75|H_L(2\pi \cdot 50)| = 2.2315 \text{ V}
\]
\[
|V_{o,H}| = |V_i(2\pi \cdot 50)||H_H(2\pi \cdot 50)| = 0.75|H_B(2\pi \cdot 50)| = 0.156 \text{ mV}
\]
\[
|V_{o,B}| = |V_i(2\pi \cdot 50)||H_B(2\pi \cdot 50)| = 0.75|H_B(2\pi \cdot 50)| = 0.246 \text{ V}
\]

For \( V_b(t) \):

\[
|V_{o,L}| = 1.25|H_L(2\pi \cdot 450)| = 0.898 \text{ V}
\]
\[
|V_{o,H}| = 1.25|H_B(2\pi \cdot 450)| = 0.230 \text{ V}
\]
\[
|V_{o,B}| = 1.25|H_B(2\pi \cdot 450)| = 1.664 \text{ V}
\]

For \( V_h(t) \):

\[
|V_{o,L}| = 2.5|H_L(2\pi \cdot 3000)| = 0.276 \text{ V}
\]
\[
|V_{o,H}| = 2.5|H_B(2\pi \cdot 3000)| = 1.952 \text{ V}
\]
\[
|V_{o,B}| = 2.5|H_B(2\pi \cdot 3000)| = 0.976 \text{ V}
\]
(c) Finally, we will get a rough idea for how the color organ will function (on paper) before we finally put those microphones to good use in the lab. Given the above definitions for $V_l$, $V_b$, and $V_h$, plot the output voltage for the low pass filter in red, bad pass filter in green, and high pass filter in blue for the following microphone output waveform for $0 \leq t \leq 4$.

$$V_{\text{mic}}(t) = \begin{cases} 
V_l(t) & \text{for } t < 1.5 \\
V_b(t) & \text{for } t \geq 1.5 \text{ and } t \leq 2 \\
V_h(t) & \text{for } t > 2
\end{cases}$$

Solution:

![Diagram showing output voltage for low pass filter in red, bad pass filter in green, and high pass filter in blue for the microphone output waveform]

Figure 19: $V_{\text{out}}$ of the three branches vs. time

6. Multiplying DAC

Previously, you analyzed one way of using a R-2R ladder to convert a digital binary number into an analog voltage. In this problem, we will examine a different way of using the R-2R ladder to make a DAC.
Let \( b_0, b_1, b_2 = \{0, 1\} \) (that is, either 1 or 0) and let \( V_{\text{ref}} = V_{\text{DD}} \). If a switch \( b_i \) is on (1), then the current in that branch with the 2R resistor is allowed to flow into the output wire. Otherwise, the current flows into the ground. Notice that with the op-amp’s negative feedback, the amount of current flowing through the 2R resistors is constant regardless of the switch states.

As before, \((b_2, b_1, b_0)\) represents a 3-bit binary (unsigned) number where each of \( b_i \) is a binary bit.

(a) Find the current \( i_{\text{out}} \) as a function of \( b_2, b_1, b_0 \).

**Solution:** Let’s look at the resistor network on the left.

If \( b_0 = 1 \), \( i_0 \) will contribute to \( i_{\text{out}} \), but if \( b_0 = 0 \), then \( i_0 \) does not contribute. The same relationship occurs with \( i_1 \) and \( i_2 \) with \( b_1 \) and \( b_2 \), respectively. In other words, we can say:

\[
i_{\text{out}} = b_0 i_0 + b_1 i_1 + b_2 i_2
\]

Using Ohm’s law we can say

\[
i_0 = \frac{V_0}{2R} \quad i_1 = \frac{V_1}{2R} \quad i_2 = \frac{V_{\text{ref}}}{2R}
\]

We can find \( V_0 \) and \( V_1 \) in terms of \( V_{\text{ref}} \) by using KCL at \( V_0 \) and \( V_1 \):

\[
\frac{V_0}{2R} + \frac{V_0}{2R} + \frac{V_0 - V_1}{R} = 0
\]

\[
\frac{V_1 - V_0}{R} + \frac{V_1}{2R} + \frac{V_1 - V_{\text{ref}}}{R} = 0
\]
Solving this system of equations, we find that

\[ V_1 = \frac{V_{\text{ref}}}{2} \]
\[ V_0 = \frac{V_1}{2} = \frac{V_{\text{ref}}}{4} \]

plugging these voltages back into the \( i_{\text{out}} \) equation we get:

\[ i_{\text{out}} = b_0 \frac{V_{\text{ref}}}{8} + b_1 \frac{V_{\text{ref}}}{4} + b_2 \frac{V_{\text{ref}}}{2} \]

(b) Finally, solve for \( V_{\text{out}} \) using the current \( i_{\text{out}} \). Express your final answer in terms of \( V_{\text{DD}} \) and the binary bits \( b_2, b_1, b_0 \). What do you notice? What is different compared to the previous R-2R DAC in homework 1, and if there is a difference, what can we do to remedy it?

**Solution:** By Ohm’s law over the final resistor \( R \), we get:

\[ V_{\text{out}} - 0 = -i_{\text{out}}R \]
\[ V_{\text{out}} = -\left( \frac{b_2 V_{\text{ref}}}{2} + \frac{b_1 V_{\text{ref}}}{4} + \frac{b_0 V_{\text{ref}}}{8} \right) \]

The output voltage is exactly the same as in homework 1 but inverted! To remedy this, you need to invert the voltage (e.g. using another op-amp). In fact, this is exactly what you did in the second DAC/ADC lab :)!

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