1. Mechanical Inverses

In each part, determine whether the inverse of \( A \) exists. If it exists, find it.

(a) \[ A = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \]

(b) \[ A = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} \]

(c) \[ A = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 0 & 4 \end{bmatrix} \]

(d) \[ A = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix} \]

Reference Definitions

Vector spaces: A vector space \( V \) is a set of elements that is ‘closed’ under vector addition and scalar multiplication and contains a zero vector. What does closed mean?

That is, if you add two vectors in \( V \), your resulting vector will still be in \( V \). If you multiply a vector in \( V \) by a scalar, your resulting vector will still be in \( V \).

More formally, a vector space \((V, F)\) is a set of vectors \( V \), a set of scalars \( F \), and two operators that satisfy the following properties:

- **Vector Addition**
  - Associative: \( \bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w} \) for any \( \bar{v}, \bar{u}, \bar{w} \in V \).
  - Commutative: \( \bar{u} + \bar{v} = \bar{v} + \bar{u} \) for any \( \bar{v}, \bar{u} \in V \).
  - Additive Identity: There exists an additive identity \( \bar{0} \in V \) such that \( \bar{v} + \bar{0} = \bar{v} \) for any \( \bar{v} \in V \).
  - Additive Inverse: For any \( \bar{v} \in V \), there exists \( -\bar{v} \in V \) such that \( \bar{v} + (-\bar{v}) = \bar{0} \). We call \( -\bar{v} \) the additive inverse of \( \bar{v} \).

- **Scalar Multiplication**
  - Associative: \( (\alpha \beta)\bar{v} = \alpha (\beta \bar{v}) \) for any \( \bar{v} \in V \), \( \alpha, \beta \in F \).
  - Multiplicative Identity: There exists 1 \( \in F \) where \( 1 \cdot \bar{v} = \bar{v} \) for any \( \bar{v} \in F \). We call 1 the multiplicative identity.
  - Distributive in vector addition: \( \alpha (\bar{u} + \bar{v}) = \alpha \bar{u} + \alpha \bar{v} \) for any \( \alpha \in F \) and \( \bar{u}, \bar{v} \in V \).
  - Distributive in scalar addition: \( (\alpha + \beta)\bar{v} = \alpha \bar{v} + \beta \bar{v} \) for any \( \alpha, \beta \in F \) and \( \bar{v} \in V \).
Subspaces: A subset $W$ of a vector space $V$ is a subspace of $V$ if the above conditions (closure under vector addition and scalar multiplication and existence of a zero vector) hold for the elements in the subspace $W$. The vector spaces we will work with most commonly are $\mathbb{R}^n$ and $\mathbb{C}^n$ as well as their subspaces.

2. Identifying a Basis

Does each of these sets of vectors describe a basis for $\mathbb{R}^3$? What about for some subspace of $\mathbb{R}^3$?

$$V_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \quad V_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \quad V_3 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

3. Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \mid \vec{v} = c \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + d \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why/why not?