1. Constructing a Basis

Let’s consider a subspace of $\mathbb{R}^3$ called $V$ which has the following property: for every vector in $V$, the first entry is equal to two times the sum of the second and third entries. That is, if \[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix} \in V, \quad a_1 = 2(a_2 + a_3).
\]

Find a basis for $V$. What is the dimension of $V$?

**Answer:**

Any vector $\vec{v}$ in $V$ is going to look as follows:

\[
\vec{v} = \begin{bmatrix} 2(a_2 + a_3) \\ a_2 \\ a_3 \end{bmatrix} = a_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}
\]

Now, we consider the set of vectors $B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. The vectors are linearly independent. Furthermore, from the above equation, any vector $\vec{v} \in V$ can be expressed as a linear combination of the vectors in $B$ (the corresponding coefficients are $a_2$ and $a_3$). This means that $V = \text{span}\{B\}$.

Therefore,

\[
B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

forms a basis for $V$.

$\text{dim}(B) = 2$ (there are two vectors in $B$), so the dimension of $V$ is 2.

2. Exploring Dimension, Linear Independence, and Basis

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimension of a vector space/subspace, and basis.

Let’s consider the vector space $\mathbb{R}^m$ and a set of $n$ vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^m$.

(a) For the first part of the problem, let $m > n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? Why/why not? What conditions would we need?

**Answer:**

No. $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ cannot form a basis for $\mathbb{R}^m$. The dimension of $\mathbb{R}^m$ is $m$, so you would need $m$ linearly independent vectors to describe the vector space. Since $n < m$, this is not possible.

(b) Let $m = n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? Why/why not? What conditions would we need?

**Answer:**

Yes, this is possible. The only condition we need is that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent. If the vectors are linearly independent, since there are $m$ of them, they will span $\mathbb{R}^m$. 
(c) Now, let $m < n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? What vector space could they form a basis for?

*Hint:* Think about whether the vectors can be linearly independent.

**Answer:**

No, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ cannot form a basis for $\mathbb{R}^m$. $\mathbb{R}^m$ will be spanned by $m$ linearly independent vectors. Any additional vectors in $\mathbb{R}^m$ must already exist in the span of the previous vectors, and are therefore linearly dependent. Since $n > m$, some of the vectors have to be linearly dependent, so they cannot form a basis.

The two regimes—one where $n > m$ and one where $n < m$—give rise to two different classes of interesting problems. You might learn more about them in upper division courses!

3. Exploring Column Spaces and Null Spaces

- The **column space** is the possible outputs of a transformation/function/linear operation. It is also the span of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of $A$? What is its dimension?

ii. What is the null space of $A$? What is its dimension?

iii. Are the column spaces of the row reduced matrix $A$ and the original matrix $A$ the same?

iv. Do the columns of $A$ form a basis for $\mathbb{R}^2$ (or $\mathbb{R}^3$ for (c))? Why or why not?

(a) \[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\]

**Answer:**

Column space: span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

The matrix is already row reduced. The column spaces of the row reduced matrix and the original matrix are the same.

Not a basis for $\mathbb{R}^2$.

(b) \[
\begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix}
\]

**Answer:**

Column space: span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

The two column spaces are not the same.

Not a basis for $\mathbb{R}^2$.

(c) \[
\begin{bmatrix}
1 & 2 \\
-1 & 1
\end{bmatrix}
\]

**Answer:**
Section 2

The two column spaces are the same as the column span \( \mathbb{R}^2 \). This is a basis for \( \mathbb{R}^2 \).

(d) \[
\begin{bmatrix}
-2 & 4 \\
3 & -6
\end{bmatrix}
\]

Answer:

Column space: \( \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \)

Null space: \( \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \)

The two column spaces are not the same.

Not a basis for \( \mathbb{R}^2 \).

(e) \[
\begin{bmatrix}
1 & 2 & 1 \\
-1 & 0 & 3 \\
0 & -1 & -2
\end{bmatrix}
\]

Answer:

Column space: \( \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} \)

Null space: \( \text{span} \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\} \)

The two column spaces are not the same.

Not a basis for \( \mathbb{R}^3 \).