1. Capacitors and Charge Conservation (with Energy!)

(a) Consider the circuit below with $C_1 = C_2 = 1 \mu F$ and an open switch. Suppose that $C_1$ is initially charged to $+1 \text{ V}$ and that $C_2$ is charged to $+2 \text{ V}$. How much charge is on $C_1$ and $C_2$? How much energy is stored in each of the capacitors? What is the total stored energy?

Answer:

$$q_1 = C_1 V_1 = 1 \mu C$$
$$q_2 = 2 \mu C$$

Energy:

$$E = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} QV$$

Note: We include a factor of $\frac{1}{2}$ because when we charge a capacitor, we do it incrementally. Not all the charge $Q$ is moved against a potential of $V$. We are basically taking $\int v(q) \, dq = \int \frac{Q}{C} \, dq = \frac{1}{2} Q^2$. Therefore, $E_1 = 0.5 \mu J$, $E_2 = 2 \mu J$, and the total energy is $2.5 \mu J$.

Note: Does it make sense that $C_2$ has more than twice the energy of $C_1$ even though it is only twice the voltage? (Yes, $C_2$ has more charge at twice the voltage.)

(b) Now the switch is closed (i.e. the capacitors are connected together.) What are the voltages across and the charges on $C_1$ and $C_2$? What is the total stored energy?

Answer:

Charge is always conserved.

Let $Q_{C_1,1}, Q_{C_2,1}$ be the charges on the capacitors after the switch is closed. There was $3 \mu C$ of total charge on the top two plates of the capacitors initially, so we must have

$$Q_{C_1,1} + Q_{C_2,1} = 3 \mu C$$

Further, the voltages on the capacitors must be the same, so:

$$\frac{Q_{C_1,1}}{C_1} = \frac{Q_{C_2,1}}{C_2}$$

Solving this system gives:

$$Q_{C_1,1} = Q_{C_2,1} = 1.5 \mu C$$
Comparing to the previous part, charge has moved from $C_2$ to $C_1$. This yields a voltage of 1.5 V. The energies are:

\[ E_1 = \frac{1}{2} (1.5\mu C)(1.5\text{ V}) = 1.125\mu J \]

\[ E_2 = \frac{1}{2} (1.5\mu C)(1.5\text{ V}) = 1.125\mu J \]

Total energy: \( E = 2.25\mu J \)

(c) Is there more or less energy than before the switch was closed? Why?

**Answer:**

The energy decreases when the switch is closed. First, the energy could not possibly be more since there is no energy input to the system. But there’s no energy output either – where did the energy go? The wires have an internal resistance that would dissipate this energy.

(d) Answer the above three questions but now with $C_1 = 2\mu F$ and $C_2 = 1\mu F$. Suppose that they are initially charged in the same way: $C_1$ is charged to $+1\text{ V}$, and $C_2$ is charged to $+2\text{ V}$.

(e) Consider the following circuit with $C_1 = 1\mu F$ and $C_2 = 3\mu F$. Suppose that both capacitors are initially uncharged (0 V).

What are the voltages across each capacitor after the switch is closed? What are the charges on each capacitor?

**Answer:**

**Solution 1:** Use equivalent capacitance:

\[ C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} \]

The series has an equivalent capacitance \( C_{eq} = \frac{3}{4}\mu F \), so \( Q_{eq} = C_{eq}V = \frac{3}{4}\mu C \). Note that this means $+\frac{3}{4}\mu C$ is on the top plate of $C_1$ (and hence the top plate of $C_2$, etc.).

The voltage across $C_1$ is $\frac{Q_{eq}}{C_1} = \frac{3}{4}\text{ V}$. The voltage across $C_2$ is $\frac{Q_{eq}}{C_2} = \frac{1}{4}\text{ V}$.

**Solution 2:** Let an unknown $+q$ charge be on the top plate of $C_1$. Then by charge conservation, $-q$ charge is on the bottom plate of $C_2$. And since conductors have no \( \vec{E} \) field, $-q$ is on the bottom plate of $C_1$, and $+q$ on the top of $C_2$.

Now, by KVL, the voltage across the series is 1 V:

\[ \frac{q}{C_1} + \frac{q}{C_2} = 1\text{ V} \]

Therefore:

\[ q = \frac{1\text{ V}}{\frac{1}{C_1} + \frac{1}{C_2}} \]

Notice that we have derived the formula for equivalent capacitance of capacitors in series.
2. Charge Sharing

Consider the circuit shown below. In phase $\phi_1$, the switches labeled $\phi_1$ are on while the switches labeled $\phi_2$ are off. In phase $\phi_2$, the switches labeled $\phi_2$ are on while the switches labeled $\phi_1$ are off.

(a) Redraw the circuit in phase $\phi_1$. Label the voltages across each capacitor and find the charge on and voltage across each capacitor as a function of $V_{in}$, $C_1$, and $C_2$. Assume the capacitors are uncharged before phase $\phi_1$.

Answer:

The two capacitors in series have a total capacitance of $\frac{C_1C_2}{C_1+C_2}$. We know that there is a voltage of $V_{in}$ across this capacitor and thus $V_{in}\frac{C_1C_2}{C_1+C_2}$ charge. The charge on $C_1$ must be equal to the charge on $C_2$.

Knowing the charge on each capacitor, we know the voltage across both. Therefore, the voltage across $C_1$ is $\frac{C_1}{C_1+C_2}V_{in}$. The voltage across $C_2$ is similarly $\frac{C_2}{C_1+C_2}V_{in}$.

(b) Redraw the circuit in phase $\phi_2$. Label the voltages across each capacitor and find the charge on and voltage across each capacitor as a function of $V_{out}$, $C_1$, and $C_2$.

Answer:

The charge on $C_1$ is simply $C_1V_{out}$. Similarly the charge on $C_2$ is $C_2V_{out}$.

(c) Find $V_{out}$ as a function of $V_{in}$, $C_1$, and $C_2$.

Answer:

We know the total charge in the system is conserved between phase $\phi_1$ and phase $\phi_2$. There was a charge of $V_{in}\frac{C_1C_2}{C_1+C_2}$ on each capacitor, so the total charge in phase $\phi_1$ was $2V_{in}\frac{C_1C_2}{C_1+C_2}$. Since we know charge is conserved, this must be equal to the total charge in phase $\phi_2$.

$$2V_{in}\frac{C_1C_2}{C_1+C_2} = (C_1+C_2)V_{out}$$
\[ V_{\text{out}} = 2 \frac{C_1 C_2}{(C_1 + C_2)^2} V_{\text{in}} \]

(d) How will the charges be distributed in phase \( \phi_2 \) if we assume \( C_1 \gg C_2 \)?

**Answer:**
We know that the capacitors are in parallel in phase \( \phi_2 \), so the voltage across both capacitors is the same. Considering that \( Q = CV \), if \( C_1 \gg C_2 \), then \( Q_1 \gg Q_2 \).

### 3. Op-Amps As Comparators

For each of the circuits shown below, plot \( V_{\text{out}} \) for \( V_{\text{in}} \) ranging from \(-10 \text{V}\) to \(10 \text{V}\) for part (a) and from \(0 \text{V}\) to \(10 \text{V}\) for part (b). Let \( A = 100 \) for your plots. Note that in real op amps, \( A \) is typically much higher (i.e. \(10^4 - 10^7\)).

(a)

![Circuit Diagram](image)

**Answer:**

\[
\begin{align*}
V_+ &= V_{\text{in}} \\
V_- &= \frac{2 \text{k}\Omega}{1 \text{k}\Omega + 2 \text{k}\Omega} V_{\text{in}} = \frac{2}{3} V_{\text{in}} \\
V_{\text{out}} &= A(V_+ - V_-) + \frac{V_+^+ - V_-^-}{2} + V_-^- \\
&= A V_{\text{in}} \left(1 - \frac{2}{3}\right) + \frac{5 - (-5)}{2} + (-5) = \frac{1}{3} A V_{\text{in}}
\end{align*}
\]

The op-amp satisfies the linear relation above for \( V_-^{-} \leq V_{\text{out}} \leq V_+^{+} \).

\[
\begin{align*}
V_-^{-} &\leq V_{\text{out}} \leq V_+^{+} \\
V_-^{-} &\leq \frac{1}{3} A V_{\text{in}} \leq V_+^{+} \\
3 \frac{V_-^{-}}{A} &\leq V_{\text{in}} \leq 3 \frac{V_+^{+}}{A} \\
3 \frac{-5 \text{V}}{100} &\leq V_{\text{in}} \leq 3 \frac{5 \text{V}}{100} \\
-0.15 \text{V} &\leq V_{\text{in}} \leq 0.15 \text{V}
\end{align*}
\]

The op-amp saturates outside of this range.
(b) [PRACTICE]

\[ V_+ = \frac{2k\Omega}{1k\Omega + 2k\Omega} V_{in} = \frac{2}{3} V_{in} \]
\[ V_- = 2V \]
\[ V_{out} = A(V_+ - V_-) + \frac{V^+_s - V^-_s}{2} + V^-_s \]
\[ = A \left( \frac{2}{3} V_{in} - 2 \right) + \frac{5 - 0}{2} + 0 \]
\[ = A \left( \frac{2}{3} V_{in} - 2 \right) + 2.5 \]

The op-amp satisfies the linear relation above for \( V^-_s \leq V_{out} \leq V^+_s \).
\[ V_S^- \leq V_{\text{out}} \leq V_S^+ \]

\[ V_S^- \leq A \left( \frac{2}{3} V_{\text{in}} - 2 \right) + 2.5 \leq V_S^+ \]

\[ \frac{3}{2} \left( \frac{V_S^- - 2.5}{A} + 2 \right) \leq V_{\text{in}} \leq \frac{3}{2} \left( \frac{V_S^+ - 2.5}{A} + 2 \right) \]

\[ \frac{3}{2} \left( \frac{-2.5 V}{100} + 2 \right) \leq V_{\text{in}} \leq \frac{3}{2} \left( \frac{5 V - 2.5 V}{100} + 2 \right) \]

\[ 2.9625 V \leq V_{\text{in}} \leq 3.0375 V \]

The op-amp saturates outside of this range.