1. Eigenvalues and Special Matrices – Visualization

An eigenvector $\vec{v}$ belonging to a square matrix $A$ is a nonzero vector that satisfies

$$A\vec{v} = \lambda \vec{v}$$

where $\lambda$ is a a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$.

The following parts don’t require knowledge about how to find eigenvalues. Answer each part by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

**Answer:**
Multiplying the identity matrix with any vector in $\mathbb{R}^n$ produces the same vector, that is, $I\vec{x} = \vec{x} = 1 \cdot \vec{x}$. Therefore, $\lambda = 1$. Since $\vec{x}$ can be any vector in $\mathbb{R}^n$, the corresponding eigenvectors are all vectors in $\mathbb{R}^n$.

(b) Does a diagonal matrix

$$
\begin{bmatrix}
  d_1 & 0 & 0 & \cdots & 0 \\
  0 & d_2 & 0 & \cdots & 0 \\
  0 & 0 & d_3 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & d_n
\end{bmatrix}
$$

in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

**Answer:**
Since the matrix is diagonal, multiplying the diagonal matrix with any standard basis vector $\vec{e}_i$ produces $d_i\vec{e}_i$, that is, $D\vec{e}_i = d_i\vec{e}_i$. Therefore, the eigenvalues are the diagonal entries $d_i$ of $D$, and the corresponding eigenvector associated with $\lambda = d_i$ is the standard basis vector $\vec{e}_i$.

(c) Does a rotation matrix in $\mathbb{R}^2$ have any eigenvalues $\lambda \in \mathbb{R}$?

**Answer:**
There are three cases:

i. Rotation by $0^\circ$ (more accurately, any integer multiple of $360^\circ$), which yields a rotation matrix $R = I$: This will have one eigenvalue of $+1$ because it doesn’t affect any vector ($R\vec{x} = \vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

ii. Rotation by $180^\circ$ (more accurately, any angle of $180^\circ + n \cdot 360^\circ$ for integer $n$), which yields a rotation matrix $R = -I$: This will have one eigenvalue of $-1$ because it “flips” any vector ($R\vec{x} = -\vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

iii. Any other rotation: there aren’t any real eigenvalues. The reason is, if there were any real eigenvalue $\lambda \in \mathbb{R}$ for a non-trivial rotation matrix, it means that we can get $R\vec{x} = \lambda \vec{x}$ for some $\vec{x} \neq \vec{0}$, which means that by rotating a vector, we scaled it. This is a contradiction (again, unless $R = I$). Refer to Figure 1 for a visualization.

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Figure 1: Rotation will never scale any non-zero vector (by a real number) unless it is rotation by an integer multiple of 360° (identity matrix) or the rotation angle is \( \theta = 180° + n \cdot 360° \) for any integer \( n \) \((-I)\).

(d) Does a reflection matrix, where the reflection is around any line passing through the origin, have any eigenvalues \( \lambda \in \mathbb{R} \)?

**Answer:**
Yes, both +1 and −1. Why? Reflecting any vector that is on the reflection axis will not affect it (eigenvalue +1). Reflecting any vector orthogonal to the reflection axis will just “flip it/negate it” (eigenvalue −1). In other words, the axis of reflection is the eigenspace associated with the eigenvalue +1 and the orthogonal space to that axis of reflection is the eigenspace associated with the eigenvalue −1. Refer to Figure 2 for a visualization.

![Reflection](image1.png)

Figure 2: Reflection will scale vectors on the reflection axis (by +1) and orthogonal to it (by −1).

(e) If a matrix \( M \) has an eigenvalue \( \lambda = 0 \), what does this say about its null space? What does this say about the solutions of the system of linear equations \( M\vec{x} = \vec{b} \)?

**Answer:**
dim(Null(M)) > 0
Mx = ̃b has no unique solution.

(f) Does the matrix \[
\begin{bmatrix}
1 & 1 \\
0 & 0 \\
\end{bmatrix}
\] have any eigenvalues \(\lambda \in \mathbb{R}\)? What are the corresponding eigenvectors?

Answer:
Note that the matrix has linearly dependent columns. Therefore, according to part (e) one eigenvalue is \(\lambda = 0\). The corresponding eigenvector, which is equivalent to the basis vector for the null space, is \[
\begin{bmatrix}
1 \\
-1
\end{bmatrix}
\]. The other eigenvalue is, by inspection, \(\lambda = 1\) with the corresponding eigenvector \[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\] because \[
\begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix} = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\].

2. Mechanical Determinants

(a) Compute the determinant of \[
\begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix}
\].

Answer:
\[
det \left( \begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix} \right) = 6
\]

(b) Compute the determinant of \[
\begin{bmatrix}
2 & 1 \\
0 & 3
\end{bmatrix}
\].

Answer:
\[
det \left( \begin{bmatrix}
2 & 1 \\
0 & 3
\end{bmatrix} \right) = 6
\]

3. Steady and Unsteady States

(a) You’re given the matrix \(M\) (below) which describes some physical system (could describe either people or water):
\[
M = \begin{bmatrix}
1 & 1 & -\frac{1}{2} \\
0 & 1 & -\frac{1}{2} \\
0 & 0 & 2
\end{bmatrix}
\]

Find the eigenspaces associated with the following eigenvalues:

i. span(\(v_1\)), associated with \(\lambda_1 = 1\)
ii. span(\(v_2\)), associated with \(\lambda_2 = 2\)
iii. span(\(v_3\)), associated with \(\lambda_3 = \frac{1}{2}\)

Answer: This is practice finding the null space.
i. $\lambda = 1$:

\[
\begin{bmatrix}
-\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & -2 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

$\vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$

ii. $\lambda = 2$

\[
\begin{bmatrix}
-\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\
0 & -1 & -2 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-3 & 0 & -3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

$\vec{v}_2 = \beta \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \beta \in \mathbb{R}$

iii. $\lambda = \frac{1}{2}$

\[
\begin{bmatrix}
0 & \frac{1}{2} & -\frac{1}{2} & 0 \\
0 & \frac{1}{2} & -2 & 0 \\
0 & 0 & \frac{3}{2} & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

$\vec{v}_3 = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \gamma \in \mathbb{R}$

(b) Define $\vec{x} = \alpha \vec{v}_1 + \beta \vec{v}_2 + \gamma \vec{v}_3$. The values $\alpha, \beta, \gamma$ are the coordinates for the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. For each of the cases in the table, determine if

\[
\lim_{n \to \infty} M^n \vec{x}
\]

converges. If it does, what does it converge to?

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>Converges?</th>
<th>$\lim_{n \to \infty} M^n \vec{x}$</th>
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Answer:
<table>
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<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>Converges?</th>
<th>$\lim_{n \to \infty} M^n \vec{x}$</th>
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