1. Proofs

(a) Prove the following statement (proved earlier in lecture): If the columns of \( A \) are linearly dependent, then \( A\vec{x} = \vec{b} \) does not have a unique solution.

**Answer:** Let’s walk through this proof step by step: we’ll start by assuming we have a matrix \( A \) with linearly dependent columns, and then we will show that this means that the system does not have a unique solution.

Since we are interested in the columns of \( A \), let’s start by explicitly defining the columns of \( A \):

\[
A = \begin{bmatrix}
\vec{a}_1 \\
\vec{a}_2 \\
\vdots \\
\vec{a}_n
\end{bmatrix}
\]

We’ve defined \( A \) to have linearly dependent columns, so by the definition of linear dependence, there exist scalars \( \alpha_1, \ldots, \alpha_n \) such that \( \alpha_1 \vec{a}_1 + \ldots + \alpha_n \vec{a}_n = \vec{0} \) where not all of the \( \alpha_i \)’s are zero. We can put these \( \alpha_i \)’s in a vector

\[
\vec{\alpha} = \begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{bmatrix}
\]

and by the definition of matrix-vector multiplication, we can compactly write the expression above:

\[
A\vec{\alpha} = \vec{0}
\]

where \( \vec{\alpha} \neq \vec{0} \).

Recall that we are trying to show that the system of equations \( A\vec{x} = \vec{b} \) does not have a unique solution. We know that systems of equations can have either zero, one, or infinite solutions. If our system of equations has zero solutions, then it cannot have a unique solution, so we don’t need to consider this case. Now let’s consider the case where we have at least one solution, \( \vec{x} \):

\[
A\vec{x} = \vec{b}
\]

\[
A\vec{x} + \vec{0} = \vec{b}
\]

\[
A\vec{x} + A\vec{\alpha} = \vec{b}
\]

\[
A(\vec{x} + \vec{\alpha}) = \vec{b}
\]

Therefore, \( \vec{x} + \vec{\alpha} \) is also a solution to the system of equations! Since both \( \vec{x} \) and \( \vec{x} + \vec{\alpha} \) are solutions, and \( \vec{\alpha} \neq \vec{0} \), the system has more than one solution. We’ve now proven the theorem.

Note that we can add any multiple of \( \vec{\alpha} \) to \( \vec{x} \) and it will still be a solution – therefore, if there is at least one solution to the system and the columns of \( A \) are linearly dependent, then there are infinite solutions.

(b) Often, when one is asked to prove something you are asked to prove something of the following nature:
• $P \implies Q$. This is read as $P$ implies $Q$.

Identify $P$ and $Q$ in the theorem you just proved above.

There are a couple of things to remember when reading these statements. First, is that the direction of implication matters.

• If you prove $P \implies Q$, this does not mean that $Q \implies P$ is also true.

Suppose someone tells you that $P \implies Q$ is true. Then you find out later that $Q$ is actually false. What can you say about $P$?

• If $P \implies Q$ and $Q$ is false, then $P$ must be false.

**Answer:** In the theorem above

• $P =$ The columns of $A$ are linearly dependent.
• $Q = A\vec{x} = \vec{b}$ does not have a unique solution.

Consider the simple example:

• $P =$ It is raining.
• $Q =$ There are clouds.
• $P \implies Q$ should be read literally as: If it is raining, then there are clouds.
• Note 1. This **does not** mean: If there are clouds, it is raining. (There could be clouds without rain).
• Note 2. This **does** however mean: If there are no clouds, it is not raining. (Because rain requires there to be clouds.)

### 2. Identifying a Basis

Does each of these sets of vectors describe a basis for $\mathbb{R}^3$? If the vectors do not form a basis for $\mathbb{R}^3$, can they be thought of as a basis for some other vector space?

$V_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$  
$V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$  
$V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

**Answer:**

• $V_1$: The vectors are linearly independent, but they are not a basis for $\mathbb{R}^3$, because you cannot construct all vectors in $\mathbb{R}^3$ using these vectors. Instead, they are a basis for some 2-dimensional subspace of $\mathbb{R}^3$.

This subspace can be described by span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

• $V_2$: Yes, the vectors are linearly independent and will form a basis for $\mathbb{R}^3$. To check that the vectors are linearly independent, you should do Gaussian Elimination of the matrix of the columns: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

Check that you can get all the way to identity, i.e. the system has a unique solution.

• $V_3$: No, $\vec{v}_2 + \vec{v}_3 = \vec{v}_1$, so the vectors are linearly dependent. Hence, they cannot form a basis for any vector space of any dimension.

### 3. Exploring Column Spaces and Null Spaces
• The **column space** is the span of the column vectors of the matrix.
• The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of $A$? What is its dimension?
ii. What is the null space of $A$? What is its dimension?
iii. Are the column spaces of the row reduced matrix $A$ and the original matrix $A$ the same?
iv. Do the columns of $A$ form a basis for $\mathbb{R}^2$? Why or why not?

(a) \[
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

**Answer:**
Column space: $\text{span}\left\{\begin{bmatrix}1 \\0\end{bmatrix}\right\}$
Null space: $\text{span}\left\{\begin{bmatrix}0 \\1\end{bmatrix}\right\}$
The matrix is already row reduced. The column spaces of the row reduced matrix and the original matrix are the same.
Not a basis for $\mathbb{R}^2$.

(b) \[
\begin{bmatrix}
0 & 1 \\
0 & 1 \\
\end{bmatrix}
\]

**Answer:**
Column space: $\text{span}\left\{\begin{bmatrix}1 \\1\end{bmatrix}\right\}$
Null space: $\text{span}\left\{\begin{bmatrix}1 \\0\end{bmatrix}\right\}$
The two column spaces are not the same.
Not a basis for $\mathbb{R}^2$.

(c) \[
\begin{bmatrix}
1 & 2 \\
-1 & 1 \\
\end{bmatrix}
\]

**Answer:**
Column space: $\mathbb{R}^2$
Null space: $\text{span}\left\{\begin{bmatrix}0 \\0\end{bmatrix}\right\}$
The two column spaces are the same as the column span $\mathbb{R}^2$.
This is a basis for $\mathbb{R}^2$.

(d) \[
\begin{bmatrix}
-2 & 4 \\
3 & -6 \\
\end{bmatrix}
\]

**Answer:**
Column space: $\text{span}\left\{\begin{bmatrix}1 \\-3/2\end{bmatrix}\right\}$
Null space: $\text{span}\left\{\begin{bmatrix}2 \\1\end{bmatrix}\right\}$
The two column spaces are not the same.
Not a basis for $\mathbb{R}^2$. 
(e) \[
\begin{bmatrix}
1 & -1 & -2 & -4 \\
1 & 1 & 3 & -3
\end{bmatrix}
\]

**Answer:**

i. The columnspace of the columns is \( \mathbb{R}^2 \). The columns of \( A \) do not form a basis for \( \mathbb{R}^2 \). This is because the columns of \( A \) are linearly dependent.

ii. The following algorithm can be used to solve for the null space of a matrix. The procedure is essentially solving the matrix-vector equation \( A\vec{x} = \vec{0} \) by performing Gaussian elimination on \( A \).

We start by performing Gaussian elimination on matrix \( A \) to get the matrix into upper-triangular form.

\[
\begin{bmatrix}
1 & -1 & -2 & -4 \\
1 & 1 & 3 & -3
\end{bmatrix} \sim \begin{bmatrix}
1 & -1 & -2 & -4 \\
0 & 2 & 5 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & -1 & -2 & -4 \\
0 & 1 & \frac{5}{2} & \frac{1}{2}
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & \frac{1}{2} & -\frac{7}{2} \\
0 & 1 & \frac{5}{2} & \frac{1}{2}
\end{bmatrix}
\]

reduced row echelon form

\[
x_1 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = 0 \\
x_2 + \frac{5}{2}x_3 + \frac{1}{2}x_4 = 0
\]

\( x_3 \) is free and \( x_4 \) is free

Now let \( x_3 = s \) and \( x_4 = t \). Then we have:

\[
x_1 + \frac{1}{2}s - \frac{7}{2}t = 0 \\
x_2 + \frac{5}{2}s + \frac{1}{2}t = 0
\]

Now writing all the unknowns \((x_1, x_2, x_3, x_4)\) in terms of the dummy variables:

\[
x_1 = -\frac{1}{2}s + \frac{7}{2}t \\
x_2 = -\frac{5}{2}s - \frac{1}{2}t \\
y = s \\
z = t
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{2}s + \frac{7}{2}t \\
-\frac{5}{2}s - \frac{1}{2}t \\
s \\
t
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{2}s \\
-\frac{5}{2}s \\
s \\
0
\end{bmatrix} + \begin{bmatrix}
\frac{7}{2}t \\
-\frac{1}{2}t \\
0 \\
t
\end{bmatrix} = s \begin{bmatrix}
-\frac{1}{2} \\
-\frac{5}{2} \\
1 \\
0
\end{bmatrix} + t \begin{bmatrix}
\frac{7}{2} \\
-\frac{1}{2} \\
0 \\
1
\end{bmatrix}
\]

So every vector in the nullspace of \( A \) can be written as follow:
Nullspace(A) = \begin{pmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}

Therefore the nullspace of A is

\[ \text{span} \left\{ \begin{pmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \right\} \]

A has a 2-dimensional null space.

iii. In this case, the column space of the row reduced matrix is also \( \mathbb{R}^2 \), but this need not be true in general.

iv. No the columns of A do not form a basis for \( \mathbb{R}^2 \).