Problem 1 (22 pts)

A system described by a linear differential equation has input $u(t)$ and output $y(t)$:

$$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 20y = -u + \frac{du}{dt}$$

[5 pts] a) Assuming zero initial conditions:

Find the transfer function $\frac{Y(s)}{U(s)} = \frac{s - 1}{s^2 + 9s + 20}$

$$Y(s) \left( s^2 + 9s + 20 \right) = U(s) \left( s - 1 \right)$$

[8 pts] b) Draw the equivalent mechanical system for this circuit, with voltage corresponding to force and current to velocity. Let $C_1 = \frac{1}{k_1}, L = M, R = B, C_2 = \frac{1}{k_2}, v_i(t) = f_i(t)$.

[9 pts] c) A nonlinear system with input $V_{in}$ and output $V_{out}$ is described by the differential equation

$$\frac{V_{out}}{R} + C\frac{dV_{out}}{dt} = e^{V_{in}-V_{out}} - 1$$

For small $V_{in}$ and $V_{out}$, find the transfer function for the linearized system:

$$V_{out}(s) = V_{in}(s) \text{ small, } V_{out} \text{ small} \Rightarrow V_{in} - V_{out} \approx 0.$$ 

$$\frac{V_{out}}{R} + C\frac{V_{out}}{R} \approx \left( V_{in} - V_{out} \right) - 1 \approx V_{in} - V_{out}$$

$$e^{V_{in} - V_{out}} \approx 1 + (V_{in} - V_{out})$$

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + \frac{1}{R} + sc} = \frac{1}{R + \frac{1}{R} + sc}$$

$$\frac{V_{out}}{R} = \frac{1}{R + \frac{1}{R} + sc}$$
Problem 2 Steady State Error (22 pts)

For the system below, let \( H(s) = \frac{8}{s+10}, G_1(s) = \frac{3t+4}{s}, \) and \( G_2(s) = \frac{2}{s+1}. \)

[5 pts] a) For \( d(t) = 0, \) and \( r(t) \) a unit step, determine \( C(s). \) \( C(s) = \)

\[
\frac{C}{R} = \frac{G_1 G_2}{1 + G_1 G_2 H} = \frac{2(s+4)}{s(s+1) + \frac{2(s+4)s}{s+10}} = \frac{2(s+4)(s+10)}{s(s+10)(s+1) + 2(s+4)s}.
\]

\[
C = \frac{C}{R} \cdot \frac{1}{s} = \frac{2}{s^2} \frac{(s+4)(s+10)}{(s+10)(s+1) + 2(s+4)}
\]

[6 pts] b) For \( d(t) = 0, \) and \( r(t) \) a unit step, find \( \lim_{t \to \infty} c(t) = \)

\[
\lim_{t \to \infty} c(t) = 5C(s) \text{ but } C(s) \text{ has } \frac{1}{s^2} \text{ term } \Rightarrow c(t) \to \infty
\]

[5 pts] c) For \( d(t) \) a unit step and \( r(t) = 0, \) determine \( C(s). \) \( C(s) = \)

\[
C = G_2 \left( \frac{D + G_1 (-H c)}{1 + G_1 G_2 H} \right)
\]

\[
C = \frac{G_2}{D + G_1 G_2 H}
\]

\[
C(s) = \frac{D(s)}{s} = \frac{2}{s} \frac{2(s+10)}{(s+10)(s+1) + 2(s+4)}
\]

[6 pts] d) For \( d(t) \) a unit step and \( r(t) = 0, \) find \( \lim_{t \to \infty} c(t) = \)

\[
\lim_{t \to \infty} c(t) = \lim_{s \to 0} sC(s) = \lim_{s \to 0} \frac{2\cdot10}{10 + 8} = \frac{20}{18} = \frac{10}{9}
\]
Problem 3. Routh-Hurwitz (15 pts)

Given open loop transfer function:

\[ G(s) = \frac{k}{(s + 3)^3} \]

and closed loop transfer function (assuming unity feedback)

\[ T(s) = \frac{k}{s^3 + 9s^2 + 27s + 27 + k} \]

[10 pts] a. Using the Routh-Hurwitz table, find the range of \( k \) for which the closed loop system is stable.

\[ -27 < k < 216 \]

\[ s^3 \quad 1 \quad 27 \quad 0 \]
\[ s^2 \quad 27+k \quad 3+k/9 \]
\[ s \quad 0 \]
\[ 0 \]

First column all positive.

\[ \#1 \quad 24-K/9 > 0 \]
\[ 24-K > K \]
\[ K < 216 \]

\[ \#2 \quad 3+K/9 > 0 \]
\[ K > -27 \]

[5 pts] b. For the positive value of \( k \) found above, find the pair of closed loop poles on the imaginary axis.

\[ s = \pm j\omega = \pm j3\sqrt{3} \]

\[ k = 216 \Rightarrow s^1 \text{ row of zeros } = 7 \] \( s^2 \text{ row for roots} \)

\[ s^2 : 9s^2 + 27 + 216 = 0 \]
\[ s^2 + 27 = 0 \]
\[ s = \pm j\sqrt{27} = \pm j3\sqrt{3} \]
Problem 4. Root Locus (17 pts)

Given open loop transfer function $G(s)$:

$$G(s) = \frac{(s + 8)}{(s + 1)(s + 2)}$$

For the root locus ($1 + kG(s) = 0$):

[2 pts] a) Determine the number of branches of the root locus =

[2 pts] b) Determine the locus of poles on the real axis

[2 pts] c) Determine the angles for each asymptote: $-180^\circ$

[6 pts] d) The break away and break in points are at $s = -1.5$ and $s = -14.5$

[5 pts] e) Sketch the root locus below using the information found above.

\[
\begin{align*}
\sigma^2 + 16\sigma + 22 &= 0 \\
\sigma &= -8 \pm \sqrt{256-22} \\
&= -8 \pm \sqrt{244} \\
&= -8 \pm 15.62 \\
36 &< 42 < 49 \\
\sqrt{42} &\approx 6.5
\end{align*}
\]
Problem 5. Root Locus Compensation (24 pts)

Given open loop transfer function $G(s)$:

$$G(s) = G_1(s) \frac{1}{(s+3)^2(s+1)^2}$$

Where $G_1(s)$ is a PD control of the form $G_1(s) = k(s + \alpha)$.

The closed loop system, using unity gain feedback and the PD controller, should have a pair of poles at $p = -1 \pm j2\sqrt{3}$.

[14 pts] a. Use the angle criteria to determine the zero location $\alpha$ for $p$ to be on the root locus. Specify the angle contributions from each open loop pole. Mark the calculated zero on the pole-zero diagram below.

$$\alpha = -1$$

$$\Delta \phi = 2 \phi \frac{1}{s+1} + 2 \phi \frac{1}{s+3} + \phi (s+\alpha)$$

$$\Delta (s+\alpha) = +120^\circ$$

$$\phi = 2(-90) + 2(-60) + \phi (s+2)$$

$$\phi = -300^\circ + \phi (s+\alpha)$$

Must equal $-180^\circ$ (or $-2\pi$).

[10 pts] b. For the determined zero location, sketch the root locus, considering real-axis segments, real-axis intercept, and asymptotes.

4 branches

$$\left( -\frac{(2\pi+1)\pi}{4-1} \right) = \frac{\pi}{3} - \frac{\pi}{2} = -\frac{\pi}{6}$$

Asymptote Intercept:

$$0 = \frac{2\phi - \phi^2}{3} = \frac{-3-3-1-(-1)}{3} = \frac{-9}{3} = -3$$

$$\phi(s) = \frac{k(s+1)}{(s+3)^2(s+1)^2}$$