Lecture #3 Phase Variable Form (Oct. 13, 2019) v. 2.0

Consider SISO LTI system with input $u(t)$ and output $y(t)$ with transfer function

$$Y(s) = \frac{b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \quad (1)$$

Introduce intermediate function $X(s)$ with

$$Y(s) = (b_4 s^4 + b_1 s^3 + b_2 s^2 + b_1 s + b_0) X(s) \quad (2)$$

and

$$X(s) = \frac{U(s)}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \quad (3)$$

Using the inverse Laplace transform on eqn. (3) we have

$$\frac{d^4 x}{dt^4} = u - a_3 \frac{d^3 x}{dt^3} - a_2 \frac{d^2 x}{dt^2} - a_1 \frac{dx}{dt} - a_0 x \quad (4)$$

This differential equation for $x(t)$ can be solved using a chain of integrators with feedback as shown here:

where $x = x_1$, $x' = x_2$ $x'' = x_3$ $x''' = x_4$ and $x'''' = \frac{dx'}{dt}$.

Referring to the block diagram, it is easy to write the state equation:

$$\dot{x} = Ax + Bu = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u(t) \quad (5)$$

**Output Equation**

Considering that by the inverse Laplace transform of eqn. (2), the output $y(t)$ is just a linear combination of the derivatives of $x(t)$, we get:

$$y(t) = b_4 \frac{d^4 x}{dt^4} + b_3 \frac{d^3 x}{dt^3} + b_2 \frac{d^2 x}{dt^2} + b_1 \frac{dx}{dt} + b_0 x \quad (6)$$

The output $y(t)$ is shown in the block diagram.

In terms of the state variable assignment we have $y(t) = b_4 x_4 + b_3 x_4 + b_2 x_3 + b_1 x_2 + b_0 x_1$. However, the output equation $y = Cx + Du$ needs to be in terms of the states, thus we need to calculate $x_4$ by using eqn. (4). Hence,

$$y(t) = b_4 (u - a_3 x_4 - a_2 x_3 - a_1 x_2 - a_0 x_1) + b_3 x_4 + b_2 x_3 + b_1 x_2 + b_0 x_1 \quad (7)$$

By combining terms, we get

$$y(t) = b_4 u + (b_3 - b_4 a_3) x_4 + (b_2 - b_4 a_2) x_3 + (b_1 - b_4 a_1) x_2 + (b_0 - b_4 a_0) x_1 \quad (8)$$

then the corrected output equation for controllable canonical form is:

$$y = Cx + Du = \begin{bmatrix} b_0 - b_4 a_0 & b_1 - b_4 a_1 & b_2 - b_4 a_2 & b_3 - b_4 a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + b_4 u(t) \quad (9)$$