• Closed book. One page, 2 sides of formula sheets. No calculators.

• There are 8 problems worth 100 points total.

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MAX = 94
MIN = 16
mean = 52.7/100
s = 17.4
median = 54

In the real world, unethical actions by engineers can cost money, careers, and lives. The penalty for unethical actions on this exam will be a grade of ‘F’ and a letter will be written for your file and to the Office of Student Conduct.

\[
\begin{align*}
\tan^{-1} \frac{1}{3} &= 26.6^\circ \\
\tan^{-1} \frac{1}{3} &= 18.4^\circ \\
\tan^{-1} \frac{1}{3} &= 14^\circ \\
\tan^{-1} \sqrt{3} &= 60^\circ \\
\tan^{-1} \frac{1}{\sqrt{3}} &= 30^\circ \\
\sin 30^\circ &= \frac{1}{2} \\
\cos 60^\circ &= \frac{\sqrt{3}}{2}
\end{align*}
\]

<table>
<thead>
<tr>
<th>(20 \log_{10} 1 = 0dB)</th>
<th>(20 \log_{10} 2 = 6dB)</th>
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<tbody>
<tr>
<td>(20 \log_{10} \sqrt{2} = 3dB)</td>
<td>(20 \log_{10} \frac{1}{2} = -6dB)</td>
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<tr>
<td>(20 \log_{10} 5 = 20dB - 6dB = 14dB)</td>
<td>(20 \log_{10} \sqrt{10} = 10dB)</td>
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<td>(1/e \approx 0.37)</td>
<td>(1/e^2 \approx 0.14)</td>
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<tr>
<td>(1/e^3 \approx 0.05)</td>
<td>(\sqrt{10} \approx 3.16)</td>
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Key.

Problem 1 (14 pts)

You are given the open-loop plant:

\[ G(s) = \frac{s + 5}{(s + 2)(s + 10)^3}. \]

For the above system, the partial root locus is shown for 5 different controller/plant combinations, \( G(s), D_2(s)G(s), \ldots, D_5(s)G(s) \). (Note: the root locus shows open-loop pole locations for \( D(s)G(s) \), and closed-loop poles for \( \frac{DG}{1+DG} \) are at end points of branches).

[5 pts] a) For each set of open-loop poles and zeros given above, choose the best corresponding open-loop Bode plot \( V, W, X, Y, \) or \( Z \) from the next page:

(i) \( G(s) \): Bode Plot \( W \)
(ii) \( D_2(s)G(s) \): Bode plot \( Z \)
(iii) \( D_3(s)G(s) \): Bode plot \( V \)
(iv) \( D_4(s)G(s) \): Bode Plot \( Y \)
(v) \( D_5(s)G(s) \): Bode Plot \( X \)
Problem 1, cont.
The open-loop Bode plots for 5 different controller/plant combinations, $D_1(s)G(s), ..., D_5(s)G(s)$ are shown below.

[4 pts] b) For the listed Bode plots, estimate the phase and gain margin:

(i) Bode plot V: phase margin $45^\circ$ (degrees) at $\omega = 8$
    Bode plot V: gain margin $10$ dB at $\omega = 12$

(ii) Bode plot Z: phase margin $20^\circ$ (degrees) at $\omega = 10$
    Bode plot Z: gain margin $5$ dB at $\omega = 18$
Problem 1, cont.

[5 pts] c) For each closed loop controller/plant with root locus as given in part a), choose the best corresponding closed-loop step response (A-E)

(i) $G(s)$: step response $\underline{E}$
(ii) $D_2(s)G(s)$: step response $\underline{A}$
(iii) $D_3(s)G(s)$: step response $\underline{D}$ ($\approx V$)
(iv) $D_4(s)G(s)$: step response $\underline{C}$
(v) $D_5(s)G(s)$: step response $\underline{B}$

$D_4$ has greater $\phi$ than $D_3$. 

---

Plot 2

Step A

Step B (For $D_5$)

Step C

Step D

Step E

---

$\omega_n \approx \frac{2\pi}{2} = \pi$
Problem 2 (14 pts)

[4 pts] a. You are given the open loop plant: \( G(s) = k\frac{s+6}{s+6} \), with \( D(s) = 1 \). Sketch Nyquist plot for \( G(s) \) with \( k = 1 \), showing clearly any encirclements.

\[
\begin{array}{c|c|c}
\text{w} & 16 \mid & 46 \\
0 & 1 & -180^\circ \\
6 & 45^\circ -135^\circ = -90^\circ \\
6\sqrt{3} & 60^\circ -120^\circ = -60^\circ \\
\infty & 90^\circ -90^\circ = 0 \\
\end{array}
\]

[2 pts] b. Find the bounds on \( k \) for the system with unity feedback to be stable.

\( k > 1 \) gives CCW encirclement of \(-1\). \( N = 1 \) CCW

\( p = 1 \) O.L.R.H.P. pole

[4 pts] c. You are given the open loop plant \( G(s) = k\frac{(s-6)(s-4)}{(s+6)(s+1)} \).

\[ |G(j\omega) = | \frac{5-6}{5+6} | \approx 0.5 \]

Sketch Nyquist plot for \( G(s) \) with \( k = 1 \), showing clearly any encirclements. Hint: phase of \( G(j\omega = 2) \) is \(+180^\circ\).

[4 pts] d. Find the bounds on \( k \) for the system with unity feedback to be stable.

with \( k = 1 \)

\( N = 1, \ p = 1, \ t = p - N = 2 \Rightarrow \) unstable

need exactly 1 CCW encirclement

\( \Rightarrow \frac{1}{4} < k < \frac{1}{2} \)
Problem 3 (16 pts)
The open-loop system is given by \( G(s) = \frac{10^4}{(s+10)^3} \), and Bode plot for \( G(s) \) is here (Fig. 3.1):

A lag controller \( D(s) = k \frac{s+a}{s+b} \) is to be designed such that the unity gain feedback system with openloop transfer function \( D(s)G(s) \) has the same steady state error as with OLTG \( G(s) \) and has a nominal (asymptotic approximation) phase margin \( \phi_m = 55^\circ \) at \( \omega = 8 \) rad s\(^{-1} \). Note \( 20 \log |G(j\omega = j8)| = 13 \text{ dB} \).

[6 pts] a. Determine gain, zero, and pole location for the lag network \( D(s) \):

\[
\text{gain } k = \frac{\beta}{\alpha}, \quad \text{zero } \alpha = -0.8, \quad \text{pole } \beta = \frac{10}{8} = 0.2
\]

\[
0.16 > \beta > -0.2
\]

\[
\delta = -13 \text{ dB}
\]

\[
20 \log \frac{\beta}{\alpha} = -13 \text{ dB}
\]

\[
20 \log \alpha = 12 \text{ dB}
\]

\[
20 \log \beta = 14 \text{ dB}
\]

[4 pts] b. Sketch the asymptotic Bode plot for the lag network \( D(s) \) alone on the plot below (Fig. 3.2):

[4 pts] c. Sketch the asymptotic Bode plot for the combined lag network and plant \( D(s)G(s) \) on the plot (3.1) above.

[2 pts] d. Mark the phase and phase margin frequency on the plot of \( D(s)G(s) \) (Fig. 3.1). Explain briefly (1 sentence) how does the actual phase margin compare to the asymptotic prediction?

The actual phase margin will be worse than predicted since \( \Delta D(j\omega_m) \) is slightly negative, but asymptotic approx is zero.
Problem 4 (8 pts)

You are given the following plant

\[ \dot{x} = Ax + Bu, \quad y = Cx \]

where \( A \) is \( N \times N \), \( u \) is scalar, \( B \) is \( N \times 1 \), \( C \) is \( 1 \times N \), and \( x \) is \( N \times 1 \). The system is observable and controllable.

[2 pts] a. Consider a controller \( u = r - Kx \) where \( r \) is a reference input, and \( K \) is \( 1 \times N \).

Determine the transfer function \( \frac{Y(s)}{R(s)} = C(sI - A + BK)^{-1} B \)

\[ x(s) = (sI - A + BK)^{-1} BR(s) \]

\[ Y(s) = C(sI - A + BK)^{-1} BR(s) \]

[2 pts] b. Consider a controller \( u = K(r - x) \) where \( r \) is a reference input, and \( K \) is \( 1 \times N \).

Determine the transfer function \( \frac{Y(s)}{R(s)} = C(sI - A + BK)^{-1} BK \)

\[ x(s) = (sI - A + BK)^{-1} BKr \]


[3 pts] d. If the same \( K \) is used in part a. and b. above, briefly explain any difference in transfer function or behavior:

Case A and B have some eigenvalues, hence some stability and dynamic response.

In Case B, input \( r \) is vector, not scalar.

In Case B, state \( x \) and output \( y \) are scaled by \( K \) compared to case A.
Problem 5. (13 pts)
Consider the following control system:

\[ x_N = \frac{\text{Sp}}{s} \]
\[ \dot{x}_N = e - r - y = r - cx \]
\[ y = A x + B (k_e x - k x) \]

[3 pts] a. Write the state and output equations for the system, in terms of \( A, B, C, K, K_e \).

\[
\begin{bmatrix}
\dot{x} \\
\dot{x}_N
\end{bmatrix} = \begin{bmatrix}
A - BK_e & B k_e \\
-C & 0
\end{bmatrix}
\begin{bmatrix}
x \\
x_N
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} r(t),
\]
\[ y = \begin{bmatrix}
C & 0
\end{bmatrix}
\begin{bmatrix}
x \\
x_N
\end{bmatrix} \tag{1} \]

[6 pts] b. Given \( C = [1 \ 0], \ B = [0 \ 1], \ A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \)
find \( K \) and \( K_e \) such that the closed loop poles are at \( s = -1, -2, -4 \).

\[
K = \begin{bmatrix} 13 & 5 \\ k_4 & k_2 \end{bmatrix}, \quad K_e = 8
\]

\[ (s + 1)(s + 2)(s + 4) = (s^2 + 3s + 2)(s + 4) = s^3 + 7s^2 + 14s + 8 \]

[4 pts] c. Show, with \( r(t) \) a unit step input, that \( e = 0 \) in steady state (with the \( K, K_e \) found above). (Hint: do not use matrix inverse.)

\[ e = r - cx_N \]
\[ \dot{x}' = 0 \Rightarrow e = 0 \]

Steady state

\[
\begin{bmatrix}
\dot{x} \\
\dot{x}_N
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
-1 & -2 & k_e
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} r
\]
\[ 0 = x_2 \\
0 = -14x_1 - 7x_2 + 8x_N \\
0 = -x_1 + 1 \quad -14x_1 + 8x_N = 0, \quad x_N = \frac{7}{4} \]
Problem 6. 13 pts

\[ y = x_1 \]
\[ \dot{x}_1 = x_2 + u \]
\[ \dot{x}_2 = -6x_1 - 5x_2 - 2u \]

Given the following system model:

\[ \dot{x} = Ax + Bu = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u(t), \quad y = Cx = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

[3 pts] a. Write the state and output equations for the system above.

[2 pts] b. Determine if the system \( A, B, C \) is controllable and observable.

\[ C = \begin{bmatrix} B \\ AB \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \text{det} = 10, \quad \text{not controllable} \]

\[ C = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{det} = 1, \quad \text{observable}. \]

[2 pts] c. Provide state equations for an observer which takes as inputs \( u(t), y(t) \), and provides an estimate of the state \( \hat{x}(t) \).

\[ \dot{\hat{x}} = A\hat{x} + Bu + LC(y - \hat{y}) = A\hat{x} + LC(y - \hat{y}) + Bu \]
\[ \hat{y} = C\hat{x} \quad \text{with} \quad LC = \begin{bmatrix} b & d \end{bmatrix} \]

[6 pts] d. Find observer gain \( L \) such that the observer has closed loop poles at \( s_1 = -10, s_2 = -10 \).

\[ (s + 10)(s + 10) = s^2 + 20s + 100 \quad \text{is char poly.} \]

\[ |\lambda I - (A - LC)| = \begin{vmatrix} \lambda + l_1 & -1 \\ +6 + l_2 & \lambda + 5 \end{vmatrix} = (\lambda + l_1)(\lambda + 5) + 6 + l_2 = \lambda^2 + (l_1 + 5)\lambda + 5l_1 + 6 + l_2 \]
\[ \Rightarrow (l_1 + 5) = 20, \quad l_1 = 15 \]
\[ 5l_1 + 6 + l_2 = 100 \quad \Rightarrow L = \begin{bmatrix} 15 \\ 19 \end{bmatrix}. \]
Problem 7 (14 pts)

[3 pts] a. Given \( G(s) = \frac{1}{s+2} \). Let \( m(t) \) be the step response of \( g(t) \), i.e. \( M(s) = \frac{1}{s(s+2)} \). Let \( x_1(t) = m(t) - m(t-T) \) where \( T \) is the sampling period. Find \( X_1(z) \) the Z transform of \( x_1(t) \).

\[
\frac{1}{s(s+2)} = \frac{1}{s} - \frac{1}{s+2} \rightarrow \frac{1}{2} (1-e^{-2T})u(t) = m(t)
\]

\[
\frac{1}{2} u(nT) \rightarrow \left\{ \begin{array}{ll}
1 & \text{if } n \geq 0 \\
0 & \text{if } n < 0
\end{array} \right.
\]

\[
M(z) = \frac{1}{2} \left[ \frac{T}{z-1} - \frac{z}{z-e^{-2T}} \right], \quad X_1(z) = M(z) - \frac{1}{2} M(z)
\]

\[
X_1(z) = \frac{1}{2} \left[ 1 - \frac{z-1}{z-e^{-2T}} \right] = \frac{1}{2} \left( \frac{1-e^{-2T}}{z-e^{-2T}} \right)
\]

[3 pts] b. Given \( x_2(t) = -2x_2(t) + u(t) \). Find the discrete time equivalent system using zero-order hold for input \( u(t) \) and sampling period \( T \): \( x_2((k+1)T) = Gx_2(kT) + Hu(kT) \).

\[
G = e^{-2T}, \quad H = \int_{0}^{T} e^{-2\lambda} d\lambda = \frac{1}{2} e^{-2T} \int_{0}^{T} = \frac{1}{2} (1-e^{-2T})
\]

\[
G = e^{-2T}, \quad H = \frac{1}{2} (1-e^{-2T})
\]

[2 pts] c. Find the \( \frac{X_2(z)}{U(z)} \) the discrete time transfer function from input \( u \) to state \( x_2 \) using the state-space form.

\[ x_2(z) = Gx_2(z) + Hu(z) \]

\[ (zI - G)x_2(z) = Hu(z) \]

\[
\frac{X_2(z)}{U(z)} = \frac{1-e^{-2T}}{2-e^{-2T}} \quad X_2(z) = \left( zI - G \right)^{-1} Hu(z)
\]

\[
\left( z - e^{-2T} \right)^{-1} \left( \frac{1}{2} (1-e^{-2T}) \right)
\]

\[
= \frac{1}{2} \frac{1-e^{-2T}}{2-e^{-2T}}
\]
Problem 7, cont.

[2 pts] d. Does \( \frac{X_2(z)}{U(z)} = X_1(z) \)? Why or why not? Yes. In part a, \( x_1(t) \) is response to zero order hold. \( \frac{g(t)}{T} \), part b also uses zero order hold over \( T \) duration.

[4 pts] e. With zero initial conditions (ZSR), \( T = 0.25 \), and a unit step input for \( x_2(kT) \), sketch \( m(t) \) and \( x_2(kT) \) on the plot below in the interval shown:

\[
\begin{align*}
\frac{m(t)}{u(t)} & = \frac{1}{2} (1 - e^{-2t})
\end{align*}
\]

\[
\begin{align*}
x_2[k+1] & = e^{-0.5} x_2[k] + \frac{1}{2} (1 - e^{-0.5})
\end{align*}
\]

\[
\begin{align*}
k & \quad x_2[k] \\
0 & \quad 0 \\
1 & \quad \frac{1}{2} (1 - e^{-0.5}) \\
& \approx \frac{1}{2} (1 - 0.5) \\
& \approx 0.25
\end{align*}
\]
Key

Problem 8 Short Answers (8 pts)

[4 pts] a. Given the discrete time system below, find \( \lim_{k \to \infty} x(k) \) for a unit step input \( u(k) = 1 \).

\[
\begin{align*}
    x(k+1) &= \begin{bmatrix} 1/6 & 1/2 \\ 2/3 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\
    \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1/6 & 1/2 \\ 2/3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
    x_1 &= \frac{x_1}{6} + \frac{x_2}{2}, \quad \frac{5}{3} x_1 = x_2 \\
    x_2 &= \frac{2 x_1}{3} + 1, \quad \frac{5}{3} x_1 = \frac{2}{3} x_1 + 1 \Rightarrow x_1 = 1, \quad x_2 = \frac{5}{3} \\
    \lim_{k \to \infty} x(k) &= \begin{bmatrix} 1/3 \\ 5/3 \end{bmatrix}.
\end{align*}
\]

Check stability:

\[
\begin{bmatrix} x_1 - \sqrt{6} & -\sqrt{6} \\ -\sqrt{2} & 1 \end{bmatrix} = \begin{bmatrix} \lambda - \frac{3}{6} & -\frac{2}{6} \\ -\frac{\sqrt{2}}{3} & \lambda \end{bmatrix} = (\lambda - \frac{3}{6})(\lambda + \frac{2}{3})
\]

Stable \( \Rightarrow \) E.U.T.,

\( x(k+1) = x(k) \).

[4 pts] b. Given \( x(t) = -10 x(t) + u(t) \). The discrete time equivalent system using zero-order hold for input \( u(t) \) and sampling period \( T \) is of the form \( x((k + 1)T) = G x(kT) + H u(kT) \).

The discrete time system has a state feedback controller \( u(kT) = \tau(kT) - 20 x(kT) \) applied.

i. Find the eigenvalue for the closed loop system: \( 3e^{-10T} - 2 \)

\[
G = e^{aT}, \quad H = \int_0^T e^{a \lambda} \beta d\lambda = \int_0^T e^{-10 \lambda} d\lambda = \frac{e^{-10 \lambda}}{-10} \Bigg|_0^T = \frac{1}{10} (1 - e^{-10T})
\]

\[
\begin{align*}
x(k+1) &= G x(k) + H (\tau - 20 x) \\
    &= \begin{bmatrix} G - H \cdot 20 \\ H \cdot \tau \end{bmatrix} x(k) \\
    &= \begin{bmatrix} e^{-10T} - 2 + 2 e^{-10T} \\ e^{-10T} \end{bmatrix} x \\
    &= 3 e^{-10T} - 2 > -1 \\
    3 e^{-10T} &> 1 \\
    e^{-10T} &> \frac{1}{3}
\end{align*}
\]

\( -10T > \ln 1/3 \)

\( T < \frac{\ln 3}{10} \).