Due at 1700, Fri. Oct. 11 in gradescope.

Note: up to 2 students may turn in a single writeup. Reading Nise 8

1. (30 pts) Root locus sketching (Nise 8.6)
   For each part below with open loop transfer function \( G(s) \) in unity gain feedback (Fig.1):
   [2] ii) Find \( j\omega \) axis intercepts if any.
   [1] v) Verify your root locus using MATLAB.

   a) \( G(s) = \frac{k(s+20)}{s(s+20)} \)
   b) \( G(s) = \frac{k(s+20)}{s^2 + 121} \)

2. (25 pts) Root locus (Nise 8.7)
   Given the unity gain feedback system in Fig. 1, where
   \[ G(s) = \frac{K(800)(s + 20)}{s(s^2 + 20s + 200)(s + 40)} \]

   [14 pts] a) Find and approximately hand sketch the root locus using RL rules 1-8 for \( k > 0 \).
   [4 pts] b) Find the range of \( K \) which makes the system stable.
   [5 pts] c) Using the second order approximation (assuming dominant 2nd order poles) find the value of \( K \) that gives \( \zeta \approx 0.30 \) for the system’s dominant closed-loop poles.
   [2 pts] e) Use MATLAB to plot the actual step response for c) and compare to 2nd order poles approximation estimate. How far off is the approximation?

3. (26 pts) Root locus (Nise 8.6, 8.9)
   Consider the unity gain feedback system in Fig. 1 with \( G(s) = \frac{k(s^2 + s - 2)}{s^2 - 2s + 8} \). Here \( -\infty < k < \infty \)
   [6 pts] b) Find the \( j\omega \) crossing using Routh-Hurwitz.
   [4 pts] c) Hand sketch the closed-loop root locus for positive and negative \( k \).
   [2 pts] d) Find the range of \( k \) for stability.

4. (19 pts) Generalized Root locus (Nise 8.8)
   This problem exams using generalized root locus for tuning of a P+D control \( u = k_p e + k_d \dot{e} \) on a second order plant. (Fig. 2 with \( D(s) = 0, H(s) = 1 \).
   Given \( \omega_n^2 = 5, 2\zeta\omega_n = 2, k = 2, k_p = 1, k_d = 1 \) and
   \[ G_1(s) = k_d s + k_p \]
   \[ G_2(s) = \frac{k \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

   For each part below, find the characteristic equation in terms of the requested parameter, approximately hand sketch the root locus using RL rules 1-6 with respect to positive values of the changing parameter, and verify with Matlab.
   [5 pts] a) Standard root locus as \( k \) changes \( (k_d = 1, k_p = 1 \) fixed).
   [5 pts] b) Generalized root locus as \( k_p \) changes \( (k_d = 1, k = 2 \) fixed).
   [5 pts] c) Generalized root locus as \( k_d \) changes \( (k_p = 1, k = 2 \) fixed).
   [4 pts] d) Consider a system with starting values of \( k = 2, k_p = 1, k_d = 1 \). Which parameter is best to tune for reducing the settling time? Which gain parameter reduces damping factor?
\( G(s) = \frac{k(s+20)}{s(s-20)} \)

1. \# of branches = 2
2. Symmetry about real axis.
3. Left of odd poles/zeros.
4. Goes from OL poles to OL zeros.
5. \( \theta_c = \frac{(2L+1)\pi}{2-1} = -\pi \) (or any odd number times \(-\pi\))

Real axis intercept is not needed.

6. \[ \sum_{i=1}^{n} \frac{1}{s + z_i} = \sum_{i=1}^{n} \frac{1}{s + p_i} \]

\[ \frac{1}{s + 20} = \frac{1}{s - 20} + \frac{1}{s} \]

\[ s^2 - 20s = s^2 + 20s + s^2 - 400 \]
\[ 0 = s^2 + 40s - 400 \]
\[ s = -20 \pm 20\sqrt{2} \]
\[ s = -48.18, 8.28 \]

7. \[ \text{Den}_{\text{ac}}(s) = s^2 - 20s + ks + 20k \]
\[ = s^2 + (k-20)s + 20k \]

\[
\begin{bmatrix}
1 & 20k \\
 k-20 & 0 \\
 20k & 0
\end{bmatrix}
\]

For stability \( k-20 > 0 \) and \( 20k > 0 \) \( \Rightarrow k > 20 \)

8. Departure from complex pairs.
b) \( G(s) = \frac{k(s+20)}{s(s+6)(s+12)} \)

1. Number of branches = 3
2. Symmetry about real axis
3. Left of odd poles/zeros
4. Goes from OL poles to OL zeros
5. \( \theta_n = \frac{(2k-1)\pi}{3} = \frac{(2k-1)\pi}{2} \)
   
   \[ \theta_n = \frac{\pi}{2}, \frac{3\pi}{2} \]

   Real axis intercept = \( \frac{\Sigma \text{poles} - \Sigma \text{zeros}}{n-m} \)
   
   \[ = \frac{(0-6-12)-(0-20)}{3-1} = 1 \]

6. \( \frac{1}{s+20} = \frac{1}{s+6} + \frac{1}{s+12} \)

\[ 6^3 + 18s^2 + 72s = s^3 + 38s^2 + 432s + 1440 + s^3 + 32s^2 + 240s + s^3 + 26s^2 + 120s \]

\[ O = 2s^3 + 78s^2 + 720s + 1440 \]

\[ s = -26.39, -9.83, -2.77 \]

7. \( \text{Den}_{\text{num}}(s) = s^3 + 18s^2 + 72s + ks + 20k \)

\[ = s^3 + 18s^2 + (72+k)s + 20k \]

<table>
<thead>
<tr>
<th>( s )</th>
<th>18 + k</th>
</tr>
</thead>
<tbody>
<tr>
<td>72 - ( \frac{k}{72} )</td>
<td>0</td>
</tr>
<tr>
<td>20k</td>
<td>0</td>
</tr>
</tbody>
</table>

72 - \( \frac{k}{72} \) > 0, 20k > 0

\[ k < 9.72 = 648 \]

\[ k \in (0, 648) \quad \text{for stability.} \]

\[ k = 0: \quad \text{Den}_{\text{num}}(s) = s^3 + 18s^2 + 72s \]

\[ = s(s+6)(s+12) \]

Poles at \( s = -6, -12 \) on \( j\omega \) axis.

\[ k = 34.11: \quad \text{Den}_{\text{num}}(s) = s^3 + 18s^2 + 720s + 12960 \]

Poles at \( s = -18, \pm 26.83 \).

8. Departure from complex pairs.
\( G(s) = \frac{K(800)(s+20)}{s(s^2+20s+200)(s+40)} \)
\( s^2+20s+200 = (s+10+10j)(s+10-10j) \)

a) # of branches = 4

Symmetry
left at odd poles/zero
OL poles \( \rightarrow \) OL zeros

\( \theta_n = \frac{-2(l+1)\pi}{4-1} = \pi, \frac{5\pi}{3}, ... \)

Re-axis intercept = \( \frac{(0 - 10 - 10 - 40) - (-20)}{3} = -40 \approx -13.3 \)

\( \frac{1}{s+20} = \frac{1}{s} + \frac{1}{s+40} + \frac{1}{s+10+10j} + \frac{1}{s+10-10j} \)

no break away points.

\( \text{Den}(s) = s^4 + 60s^3 + 1000s^2 + (8000+800K)s + 16000K \)

\[
\begin{array}{ccc}
1 & 1000 & 16000K \\
60 & (8000+800K) & 0 \\
\frac{2000}{3} & \frac{-400K}{3} & 16000K \\
\frac{400}{K} & \frac{s+35K-650}{K-65} & 0 \\
16000K & 0 & \frac{s+35K-650}{K-65} = 0 \text{ or } 16000K = 0
\end{array}
\]

\( K = 13.4233 \quad K = 0 \)

\( \text{OL pole at origin} \)

\( \theta_{4\tau} = -180^\circ \quad \theta_{4\tau} = -71.5351^\circ \)

angle of departure: \( (s = -10-10j) \)
\( \text{atan2}(10, -10) - \text{atan2}(20, 0) - \text{atan2}(30, 10) + \text{atan}(10,10) + \theta_{4\tau} = -180^\circ \)

\( \theta_{4\tau} = 71.5351^\circ \)

b) \( K \in [0, 13.42) \)
c) \( \zeta = 0.3 \quad \Rightarrow \quad \theta = 72.54^\circ \)

\( s = a + bj = a + \text{atan}(72.54^\circ)j \)

Rest is done numerically:

```matlab
sym s;
zeta = 0.3;
K = 4.38;
foundK = false;
while ~foundK
    disp(K);
    K = K-0.001;
    den = s*(s^2+20s+200)*(s+40)+K*800*(s+20);
    poles = double(solve(den == 0,s));
    for ii = 1:length(poles)
        if imag(poles(ii)) < 0
            if abs(real(poles(ii))*tan(acos(zeta))) <= abs(imag(poles(ii)));
                foundK = true;
            end
        else
            continue;
        end
    end
end
```

\( \Rightarrow K = 4.381 \)

\( \text{poles: } -10.89 \quad -3.73 \pm 11.86j \quad -41.65 \)
d) \[ G_{\text{act}} = K \cdot \frac{1}{s^3 + 0.1s^2 + 1} \]

```
g_act = tf([800 14000],[1 20 200 0]) \cdot tf(1,[1 40]);
step(g_act/(1+g_act));
hold on;
g_app = zpk([],pole(2:3),pole(2)*pole(3));
step(g_app);
legend('Actual','Approximate');
```

The approximation is pretty close to the actual system, most of the deviation comes from the pole going from 0 to -20 on the real axis. For \( K = 4.381 \), the pole is only at -10.89 (only \(~3\times\) bigger than the real part of our dominant poles).

Actual root locus plot: (part a)
\[ G(s) = \frac{k(s^2 + s - 2)}{s^3 - 2s + 8} = \frac{k(s + 2)(s - 1)}{(s - 2)(s + 4)} \]

a) \( k > 0 \)

\[ \text{real axis segments: } (-\infty, -4], [-2, 1], [2, \infty) \]

\[ \text{break away/in phs: } \frac{1}{s+2} + \frac{1}{s-1} = \frac{1}{s-2} + \frac{1}{s+4} \]

\[ \sigma^2 + \sigma - 10\sigma + 8 = \sigma^2 + 4 \sigma^2 - 9 \sigma - 16 = \sigma^2 + 5 \sigma + 2 \sigma - 8 + \sigma^2 - \sigma^2 - 4 \sigma + 4 \]

\[ 0 = -2 \sigma^2 + 12 \sigma + 4 \]

\[ \sigma = 0.3246, 12.3246 \]

\[ \text{angle of departure: } N/A \]

\[ \text{angle of approach: } N/A \]

b) \( \text{Den}(s) = (k-1)s^3 + (k-2)s^2 + 8 - 2k \)

\[
\begin{array}{c|cc}
 k-1 & 8 - 2k \\
 k-2 & 0 \\
 8 - 2k & \\
\end{array}
\]

\[ (k-1 > 0 \land k-2 > 0 \land 8 - 2k > 0) \cup (k-1 < 0 \land k-2 < 0 \land 8 - 2k < 0) \]

\[ (k > 1 \land k > 2 \land k < 4) \cup (k < 1 \land k < 2 \land k > 4) \]

\[ (k > 2 \land k < 4) \cup \emptyset \]

\[ k = 2 : s^2 + 4 = 0 \Rightarrow \text{crosses } \emptyset \pm 2j \]

\[ k = 4 : 3s^2 + 2s = 0 \Rightarrow \text{crosses } \emptyset \]

c) \( k \in [0, \infty) \)

\[ k \in (-\infty, 0] \]

d) From part 'b') for stability \( k \in [2, 4] \)
4. \[ G_s(s) = k_d s + k_p \quad G_d(s) = \frac{k_d}{s^2 + 2s + 5} \]

a) \( k_d, k_p = 1 \)

\[
G_{ns}(s) = k \frac{5s + 5}{s^2 + 2s + 5} = k \frac{5(s+1)}{(s+1+2\delta)(s+1-2\delta)}
\]

\[
\frac{1}{s+1} = \frac{1}{\delta+1+2\delta} + \frac{1}{\delta+1-2\delta}
\]

\[
\delta^2 + 2\delta + 5 = 2\delta^2 + 4\delta + 1
\]

\[ O = \delta^2 + 2\delta - 4 \]

\[ \delta = -3.2361, 1.2361 \]

Depth angle: \[ \theta_{dep} + 90^\circ - 90^\circ = 180^\circ \]

\[ \theta_{dep} = 180^\circ \]

b) \( k_d = 1, \ k = 2 \)

\[
1 + \frac{10(s + k)}{s^2 + 2s + 5} \rightarrow s^2 + 2s + 5 + 10s + 10k \rightarrow 1 + \frac{10k}{s^2 + 12s + 5}
\]

Poles: \[ \pm \frac{12 \pm \sqrt{144 - 20}}{2} = -6 \pm 3.74i \]

Break away points: \[ O = \frac{1}{\delta + 11.57} + \frac{1}{\delta - 0.43} \]

\[ \delta = -6 \]

Asymptotes: \[ \theta_a = \frac{-3(2\pi + \pi)}{2} \]

\[ \theta_a = \frac{\pi}{2}, \frac{3\pi}{2} \]

Real axis intercept: \[ \frac{(-11.57) + (0.43)}{2} = -6 \]

c) \( k_p = 1, \ k = 2 \)

\[
1 + \frac{10(k_2 + 1)}{s^2 + 2s + 5} \rightarrow s^2 + 2s + 5 + 10k_2 + 10 \rightarrow 1 + \frac{10k_2}{s^2 + 2s + 15}
\]

Poles: \[ \pm \frac{6 \pm \sqrt{36 - 60}}{2} = -1 \pm 3.74i \]

Break away points: \[ O = \frac{1}{\delta + 1 + 3.74i} + \frac{1}{\delta + 1 - 3.74i} \]

\[ \delta^2 + 2\delta + 15 = 2\delta^2 + 2\delta \]

\[ O = \delta^2 - 15 \]

\[ \delta = \pm \sqrt{15} = \pm 3.87 \]

Depth angles: \[ \theta_{dep} + 90^\circ - \sin \left(\frac{3.74}{1}\right) = 180^\circ \]

\[ \theta_{dep} = 194.4^\circ \]

d) Settling time can be estimated from \( T_s = \frac{4}{\omega_n \delta} \), so poles ideally would be close to the real axis (large \( \delta \)) and far from the origin (large \( \omega_n \)). Therefore, tuning \( k_p \) would achieve the smallest \( T_s \). \( k_d \) is the only tuning parameter that reduces damping factor.
a) $\text{code:} \quad \text{rlocus}([5, 5], [1, 2, 5]);$

b) $\text{code:} \quad \text{rlocus}([10, 1], [1, 2, 5]);$

c) $\text{code:} \quad \text{rlocus}([10, 0], [1, 2, 15]);$