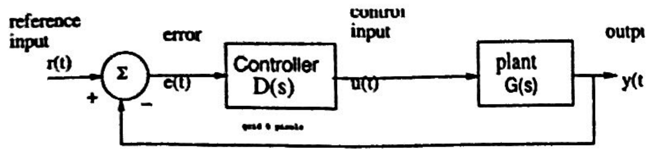
Problem 1 (22 pts)



You are given the open-loop plant:

$$G(s) = \frac{408(s+0.5)}{(s+3)(s^2+2s+17)}. \Rightarrow \text{DC gain: } G(0) = \frac{408(0.5)}{(3)(17)} = \frac{204}{51} = 4$$

For the above system, the partial root locus is shown for 5 different controller/plant combinations, $G(s), D_2(s)G(s), ..., D_5(s)G(s)$. (Note: the root locus shows open-loop pole locations for $D(s)G(s)$, and closed-loop poles for $\frac{DG}{1+DG}$ are at end points of branches).

- $3p, 1z \Rightarrow -180°$
- DC gain $= 4$ OR $20\log(4) = 40\log(2) = 40(\frac{3}{10}) = 12\,dB$

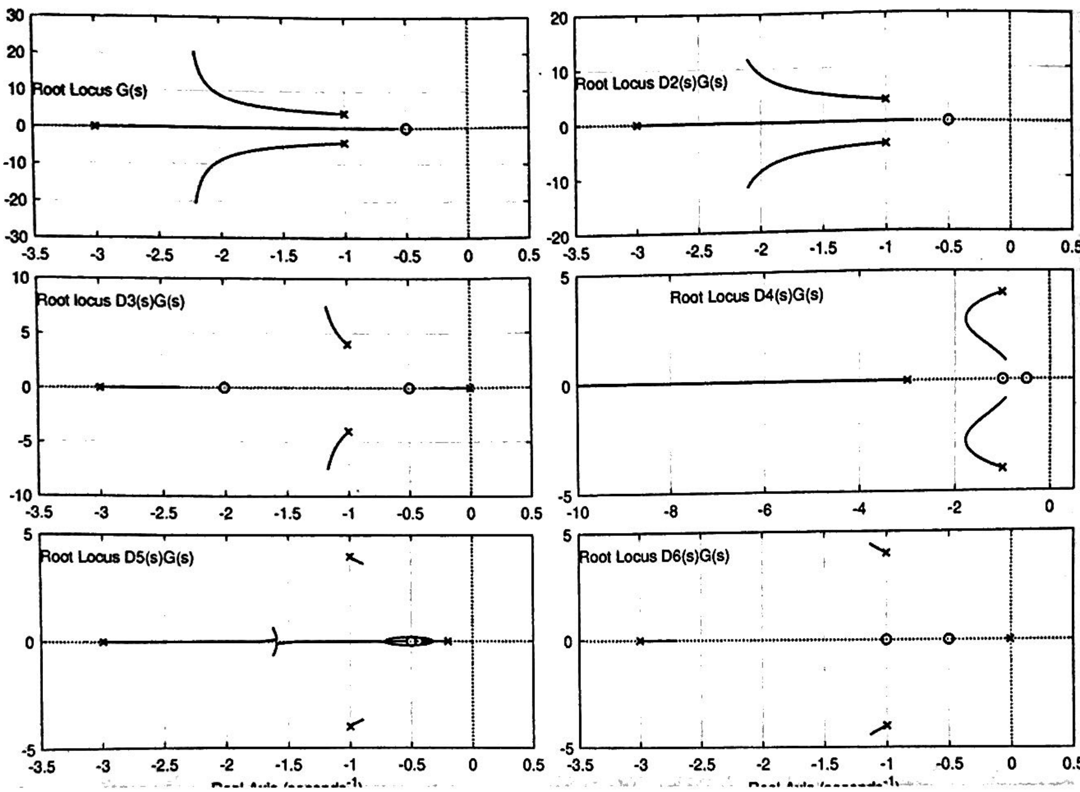
- $3p, 1z \Rightarrow -180°$
- Open-loop poles identical: $D_2(s)$ must be scaling factor

- $4p, 2z \Rightarrow -180°$
- pole @ $s=0$
- zero @ 2

- $3p, 2z \Rightarrow -90°$

- $5p, 1z \Rightarrow -360°$

- $4p, 2z \Rightarrow -180°$
- pole @ $s=0$
- zero @ 0.5



[6 pts] a) For each set of open-loop poles and zeros given above, choose the best corresponding open-loop Bode plot U,V,W,X,Y, or Z from the next page.

(i) $G(s)$: Bode plot Z DC gain ≈ 12 dB

(ii) $D_2(s)G(s)$: Bode plot W identical to $G(s)$ except for scaling of magnitude plot

(iii) $D_3(s)G(s)$: Bode plot U integral term + zero @ 2 rad/sec

(iv) $D_4(s)G(s)$: Bode Plot Y $-90°$

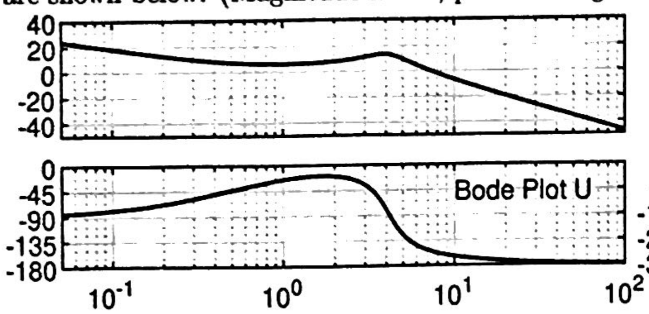
(v) $D_5(s)G(s)$: Bode Plot V $-360°$

(vi) $D_6(s)G(s)$: Bode Plot X integral term + zero @ 0.5 rad/sec

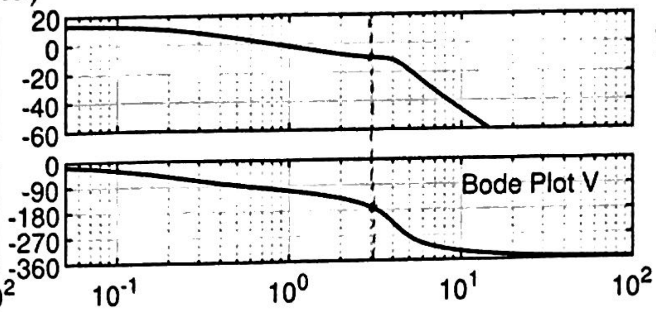
Problem 1, cont.

The open-loop Bode plots for 6 different controller/plant combinations, $D_1(s)G(s), ..., D_6(s)G(s)$ are shown below. (Magnitude in dB, phase in degrees.)

• pole @ s=0
• zero ≈ 2
$D_3(s)G(s)$

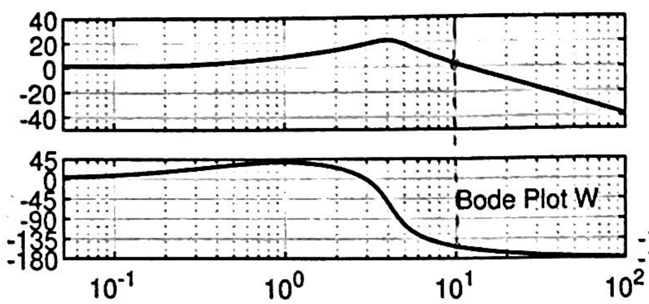


$D_5(s)\,G(s)$

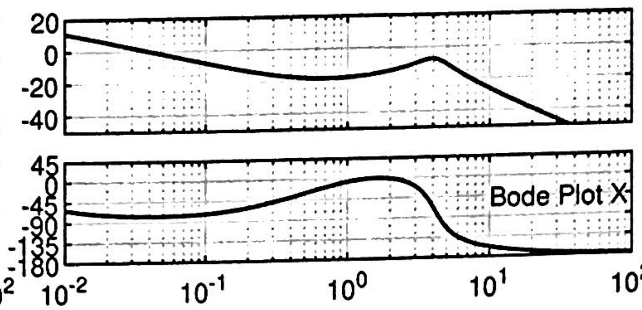


-360°

• identical Bode plot except for scaled mag. plot as Z
$D_2(s)G(s)$

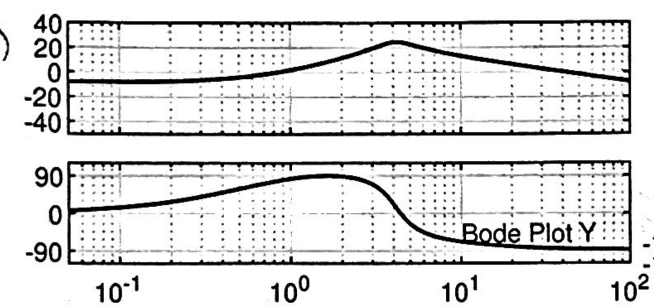


• pole @ s=0
• zero ≈ 0.5
$D_6(s)G(s)$

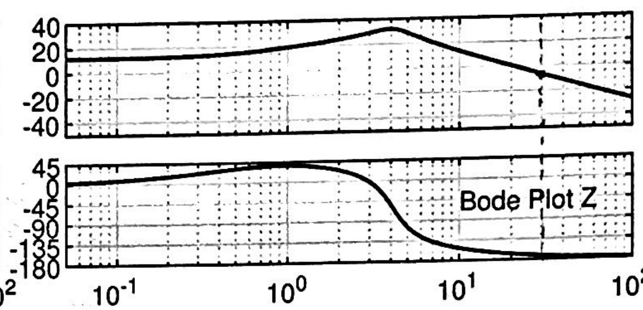


$D_4(s)G(s)$

-90°



• DC gain = 12dB
$G(s)$



[10 pts] b) For the Bode plots above:

(i) [2 pt] Which closed-loop system would have the least steady state error for a step input?
Bode plot: U
Briefly explain why: highest DC gain $\Rightarrow e_{ss} = 1 - y_{ss} = 1 - \lim_{s\to 0} s\left(\frac{1}{s}\right)\frac{D(s)G(s)}{1+D(s)G(s)} \Rightarrow$ for higher $|D(0)G(0)|$, $\lim_{s\to 0}\frac{D(s)G(s)}{1+D(s)G(s)} \to 1$

(ii) [1 pt] Which closed-loop system would have the greatest steady state error for a step input?
Bode plot: Y

(iii) [2 pt] Bode plot W: phase margin 22.5 (degrees) at $\omega =$ 10 rad/sec

(iv) [1 pt] Estimate damping factor for Bode plot W. $\zeta \approx$ 0.225 (refer to Fig. 10.48) $\Rightarrow \frac{\Phi_{PM}}{100}$

(v) [2 pt] Bode plot V: gain margin 10 dB at $\omega =$ 3

(vi) [2 pt] Estimate the closed-loop bandwidth (that is the frequency for which the closed loop system has a response of -3dB.) for the open loop response Z.
closed-loop bandwidth = 30 (rad/s) (refer to Fig. 10.49)

When Bode Plot Z has a magnitude of ≈ -7 dB, the corresponding phase is ≈ -170°

This approximately lies on the graph of Fig. 10.49, so we can estimate that @ frequency w=30 rad/sec, the closed-loop magnitude = -3dB.

Problem 1, cont.

[6 pts] c) For each closed loop controller/plant with root locus as given in part a), choose the best corresponding closed-loop step response (A-F). (Note: dashed line shows final value.)

(i) $G(s)$: step response B Imaginary poles @ $(-2.25 \pm 20j) \Rightarrow$ highest $\omega_d \Rightarrow$ lowest $T_p = \frac{\pi}{\omega_d}$

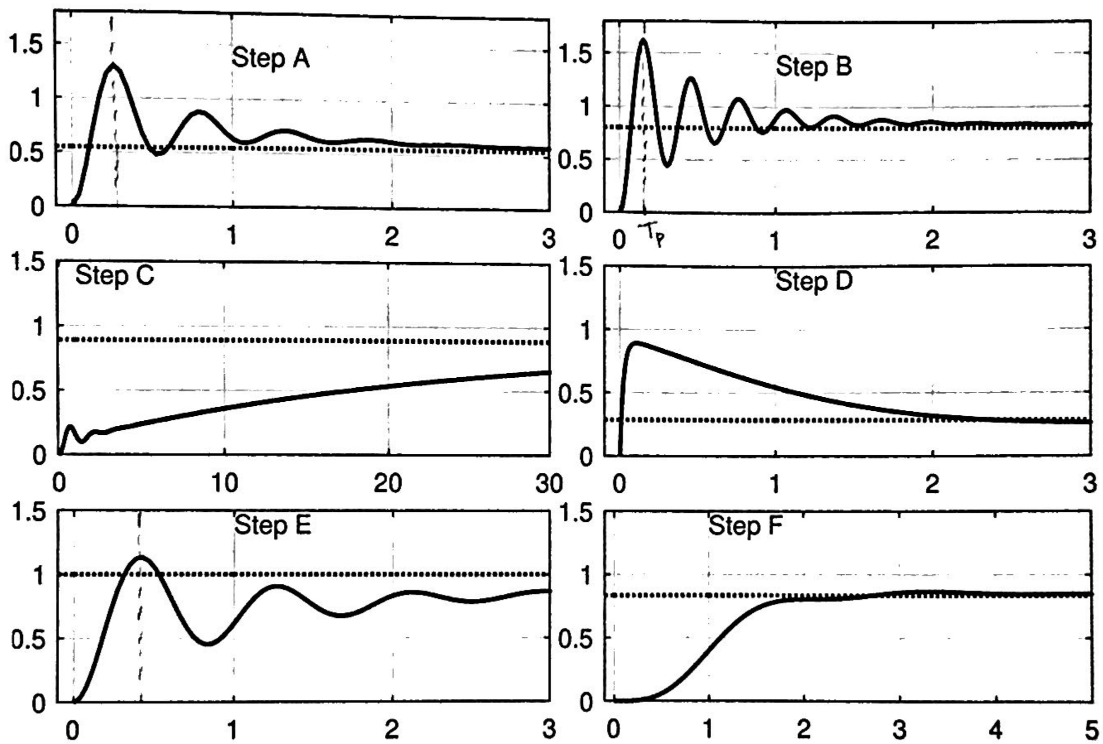
(ii) $D_2(s)G(s)$: step response A Imaginary poles @ $(-2.15 \pm 12j) \Rightarrow$ similar envelope as (i), greater T_p

(iii) $D_3(s)G(s)$: step response E Associated w/ lowest S.S.E

(iv) $D_4(s)G(s)$: step response D Associated w/ highest S.S.E, imag poles $\Rightarrow$ lowest $\omega_d \Rightarrow$ least oscillatory

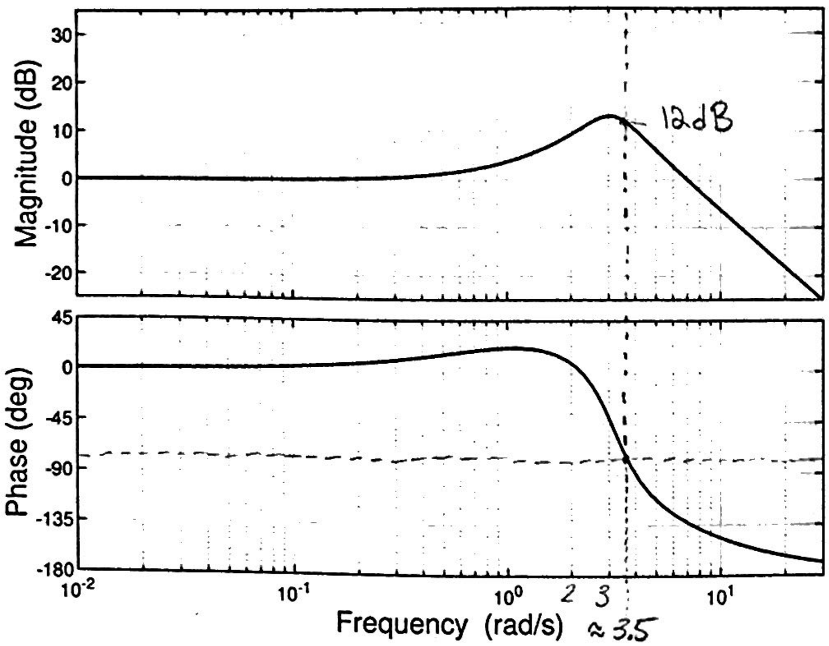
(v) $D_5(s)G(s)$: step response F Imaginary poles @ $(-0.8 \pm 4j) + (-1.65 \pm 0.5j) \Rightarrow$ high damping

(vi) $D_6(s)G(s)$: step response C Imaginary poles @ $(-1.15 \pm 4.5j)$, pole @ $s=0$ so low SSE



Problem 3 (14 pts)

The open-loop system is given by $G(s) = \frac{50(s+1)}{(s+5)(s^2+2s+10)}$, and Bode plot for $G(s)$ is here:



$K_p = 10 = D(0)G(0) = k\left(\frac{\alpha}{\beta}\right)\frac{50(1)}{(5)(10)}$

- $10 = k\left(\frac{\alpha}{\beta}\right) \Rightarrow k = 10\left(\frac{\beta}{\alpha}\right)$
- DC gain from compensator will be 10 = 20dB
- @ $\omega = 3.5$ rad/sec, need compensator to reduce gain by $(12+20) = 32$ dB
- 32, -20dB/dec, β, α $\quad 10^{32/20} = \frac{\alpha}{\beta}$
  $\Rightarrow \frac{\alpha}{\beta} = 10^{1.6} = 10 \cdot 10^{0.6}$
- $2\log_{10}(2) = 0.3 \Rightarrow \log_{10}(4) = 0.6$
  $\Rightarrow \frac{\alpha}{\beta} = 10 \cdot (4) = 40$

A lag controller $D(s) = k\frac{s+\alpha}{s+\beta}$ is to be designed such that the unity gain feedback system with openloop transfer function $D(s)G(s)$ has static error constant $K_p = 10$. $D(s)G(s)$ should have a nominal (using provided Bode diagram) phase margin $\phi_m \approx 90°$. $+10° = 100° \Rightarrow$ phase of $-80°$ has $\phi_m = 100°$

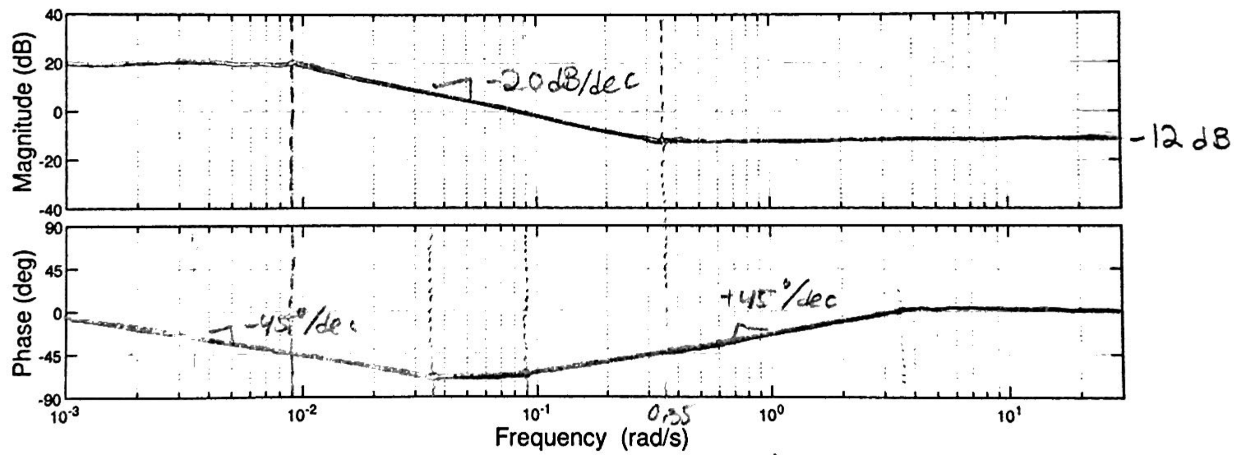
[2 pts]. a. Following the lag compensation procedure, what is the chosen phase margin frequency for the compensated system? $\quad \omega_{pm} =$ 3.5 rad s^{-1}.

[6 pts] b. Determine gain, zero, and pole location for the lag network $D(s)$:

gain $k =$ 0.25 $\quad$ zero: $\alpha =$ 0.35 $\quad$ pole: $\beta =$ $0.00875 \approx 0.009$

$10\left(\frac{\beta}{\alpha}\right) = \frac{10}{40}$ $\quad$ $\frac{\omega_{pm}}{10}$ $\quad$ $40 = \frac{\alpha}{\beta} \Rightarrow \beta = \frac{\alpha}{40}$

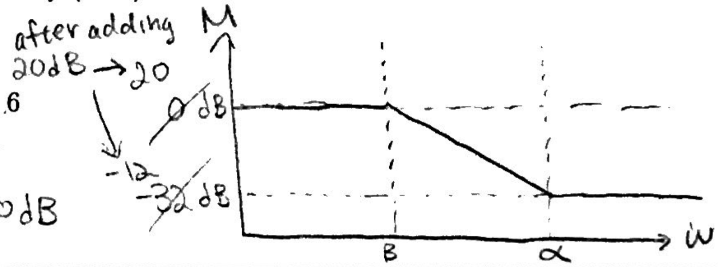
[6 pts] b. Sketch the asymptotic Bode plot for the lag network $D(s)$ *only* on the plot below:



$D(s) = k\frac{(s+\alpha)}{(s+\beta)}$

$D(0) = k\frac{\alpha}{\beta} = 10 \Rightarrow 20\log 10 = 20$

so need to shift magnitude by 20dB



Problem 4 (22 pts)

You are given the following

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}u(t), \quad y = [1 \ \ 1]\,\mathbf{x} \tag{1}$$

[2 pts] a) Determine if the system in eqn. (1) is controllable and observable.

$\mathcal{C} = \begin{bmatrix} 1 & -1 \\ 3 & -15 \end{bmatrix}$ controllable because $rk(\mathcal{C}) = 2$

$\mathcal{O} = \begin{bmatrix} 1 & 1 \\ -1 & -5 \end{bmatrix}$ observable because $rk(\mathcal{O}) = 2$

[2 pts] b) Find the transfer function for the system in eqn. (1)

$$\frac{Y(s)}{U(s)} = \frac{4(s+2)}{(s+1)(s+5)}$$

$$C(sI-A)^{-1}B = [1 \ \ 1]\begin{bmatrix} s+1 & 0 \\ 0 & s+5 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= [1 \ \ 1]\left(\frac{1}{(s+1)(s+5)}\right)\begin{bmatrix} s+5 & 0 \\ 0 & s+1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \frac{s+5+3(s+1)}{(s+1)(s+5)} = \frac{4s+8}{(s+1)(s+5)}$$

[2 pts] c) Find the equivalent system to eqn. (1) in phase variable form:

$$\dot{\mathbf{z}} = \bar{A}\mathbf{z} + \bar{B}u = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t), \quad y = [8 \ \ 4]\,\mathbf{z} \tag{2}$$

$$\frac{Y(s)}{U(s)} = \frac{4s+8}{s^2+6s+5}$$

[4 pts] d) Find the transformation P such that $\bar{A} = P^{-1}AP$ is in phase variable form.

$$P = \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}$$

$$P = [B \ \ AB][\bar{B} \ \ \bar{A}\bar{B}]^{-1}$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & -15 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -6 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & -15 \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}$$

Problem 4, cont.

[4 pts] e) State feedback with feedback gain $\mathbf{K_z}$ is applied to the system in phase variable form (eqn. 2) such that $u = r - \mathbf{K_z z}$. Given $\mathbf{K_z}$, determine the equivalent gain $\mathbf{K_x}$ for the system in eqn. (1) to have the same response with input $u = r - \mathbf{K_x x}$.

$$Px = z$$

$$u = r - K_z z = r - \underbrace{K_z P}_{K_x} x$$

$$\boxed{K_x = K_z P}$$

[4 pts] f) Find e^{At} and $e^{\bar{A}t}$. (Hint: use similarity transform):

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-5t} \end{bmatrix} \qquad e^{\bar{A}t} = \begin{bmatrix} \frac{1}{4}(5e^{-t} - e^{-5t}) & \frac{1}{4}(e^{-t} - e^{-5t}) \\ \frac{1}{4}(-5e^{-t} + 5e^{-5t}) & \frac{1}{4}(-e^{-t} + 5e^{-5t}) \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix} \quad \bar{A} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} \quad e^{\bar{A}t} = e^{P^{-1}APt} = P^{-1}e^{At}P = \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-5t} \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}$$

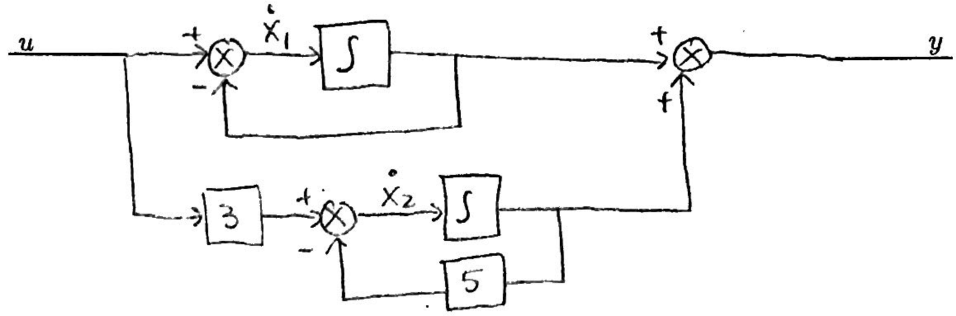
[2 pts] g) Draw a block diagram representation using integrator, scale, and sum components for the system in eqn. (1)

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u(t)$$

$\dot{x}_1 = -x_1 + u$

$\dot{x}_2 = -5x_2 + 3u$

$y = x_1 + x_2$



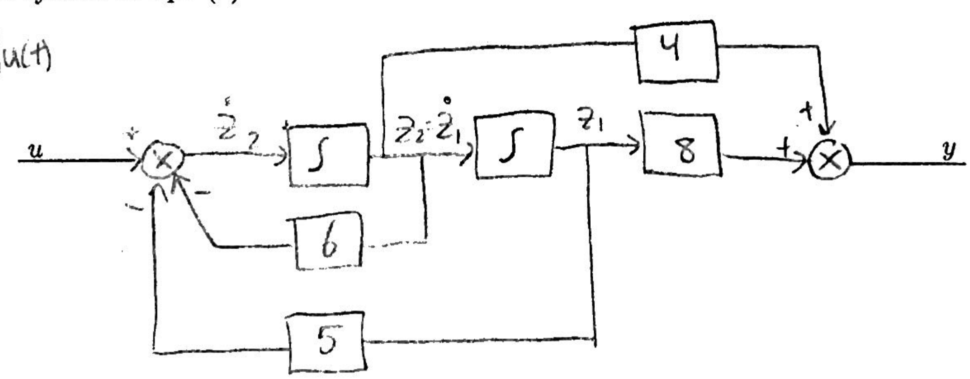
[2 pts] h) Draw a block diagram representation using integrator, scale, and sum components for the system in eqn. (2)

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$\dot{z}_1 = z_2$

$\dot{z}_2 = -5z_1 - 6z_2 + u$

$y = 8z_1 + 4z_2$



Problem 5 (12 pts)

[2 pts] a. Given the following system model:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \qquad y = C\mathbf{x}$$

Provide state equations for an observer which takes as inputs $u(t), y(t)$, and provides an estimate of the state $\hat{\mathbf{x}}(t)$.

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$
$$= A\hat{x} + Bu + LC(x - \hat{x})$$
$$= (A - LC)\hat{x} + Bu + LCx$$
$$\boxed{\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly}$$

[2 pts] b. If error e is defined as $\mathbf{e}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$, derive the error equations.

$$\dot{e} = \dot{\hat{x}} - \dot{x} = (A - LC)\hat{x} + Bu + LCx - (Ax + Bu)$$
$$= (A - LC)\hat{x} + Bu - (A - LC)x - Bu$$
$$= (A - LC)\hat{x} - (A - LC)x$$
$$\Rightarrow \boxed{\dot{e} = (A - LC)e}$$

[2 pts] c. Now consider state feedback control, with reference input r, using the state estimate from observer, $u = r - K\hat{\mathbf{x}}$. Derive the combined state equations:

$$\dot{x} = Ax + B(r - K\hat{x})$$
$$= Ax + B(r - K(e + x))$$
$$= (A - BK)x + Br - BKe$$
$$\dot{e} = (A - LC)e$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \left[\begin{array}{c|c} A-BK & -BK \\ \hline 0 & A-LC \end{array}\right] \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

[2 pts] d. Given the following system model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -14 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t), \qquad y = C\mathbf{x} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find observer gain L such that the observer has closed loop poles at $s_1 = -8, s_2 = -6$.

desired characteristic equation: $(s+8)(s+6) = s^2 + 14s + 48$

$$\det(sI - (A - LC)) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -9-L_1 & 1 \\ -14-L_2 & 0 \end{bmatrix}\right)$$

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 34 \end{bmatrix}$$

$$= \det\left(\begin{bmatrix} s+9+L_1 & -1 \\ 14+L_2 & s \end{bmatrix}\right) = s^2 + (9+L_1)s + (14+L_2)$$

$L_1 = 5 \qquad L_2 = 34$

Problem 5, continued (12 pts) $A = \begin{bmatrix} -9 & 1 \\ -14 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $C = [1 \ 0]$

[2 pts] e. Let state feedback gain $K = [-7 \ \ 3.5]$ and let observer gain $L = [2 \ \ 16]^T$. Find the eigenvalues for the combined system.

Controller: $A - BK = \begin{bmatrix} -9 & 1 \\ -14-2k_1 & -2k_1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ 0 & -7 \end{bmatrix} \Rightarrow \det\begin{pmatrix} s+9 & -1 \\ 0 & s+7 \end{pmatrix} = (s+7)(s+9)$

Observer: $A - LC = \begin{bmatrix} -9-l_1 & 1 \\ -14-l_2 & 0 \end{bmatrix} = \begin{bmatrix} -11 & 1 \\ -30 & 0 \end{bmatrix} \Rightarrow \det\begin{pmatrix} s+11 & -1 \\ 30 & s \end{pmatrix} = s^2+11s+30$

$= (s+5)(s+6)$

$\boxed{\text{Eigenvalues} = -7, -9, -5, -6}$

[2 pts] f. Consider an initial condition $\mathbf{x}(0) = [5 \ \ 10]^T$ and $\hat{\mathbf{x}}(0) = [0 \ \ 0]^T$ with $r(t) = 0$. Briefly compare the expected zero-input response for different control strategies:

Case I: $u = -K\hat{\mathbf{x}}$

and

Case II: $u = -K\mathbf{x}$.

Case II:

$u = -Kx$

$\dot{x} = Ax + B(-Kx) = (A - BK)x$

$x(t) = e^{(A-BK)t}\begin{bmatrix} 5 \\ 10 \end{bmatrix}$ with $\lambda = \{-7, -9\}$

Case I: $\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A-BK & -BK \\ 0 & A-LC \end{bmatrix}\begin{bmatrix} x(0) \\ e(0) \end{bmatrix}$

$e = \hat{x} - x$

$\begin{bmatrix} -5 \\ -10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

$\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = e^{\tilde{A}t}\begin{bmatrix} 5 \\ 10 \\ -5 \\ -10 \end{bmatrix}$ with $\lambda = \{-7, -9, -5, -6\}$

Since $x(t) \to 0$ depends on $e(t)$ also going to 0, and $e(t)$ is slower due to slower eigenvalues, the response with observer will be slower.

The observer feedback system also has slower initial response:

$\dot{x}(0) = (A - BK)x(0) - BK(e(0))$

$= Ax(0) - \cancel{BKx(0)} + \cancel{BKx(0)}$ since $e(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - x(0)$

$= A(x(0))$

but $\mathrm{eig}(A) = \begin{vmatrix} \lambda+9 & -1 \\ 14 & \lambda \end{vmatrix} = \lambda^2 + 9\lambda + 14 \Rightarrow \lambda = \{-2, -7\}$

slower eigenvalue than with $A-BK$.

Problem 6 (20 pts)

Given the following system model:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u(t), \qquad y = C\mathbf{x} = [1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note that:

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-2t} & e^{-2t} \end{bmatrix} \quad \text{for } t \geq 0. \tag{3}$$

[4 pts] a. The continuous time system in eqn. (3) is converted to discrete time using a zero-order hold with sample time $T = \ln 2$ seconds. Find G, H, C for the difference equation for $x(n)$:

$$\mathbf{x}(n+1) = G\mathbf{x}(n) + Hu(n) \qquad y(n) = C\mathbf{x}(n) \tag{4}$$

$$G = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad H = \begin{bmatrix} 1 \\ \frac{11}{8} \end{bmatrix}$$

$C = [\ 1 \quad 1\ ]$

$$G(T) \equiv e^{AT} \Rightarrow e^{-T} = e^{-\ln 2} = \frac{1}{2}$$

$$e^{-2T} = e^{-2\ln 2} = e^{-\ln 4} = \frac{1}{4}$$

$$H(T) \equiv \left(\int_0^T e^{A\lambda} d\lambda\right) B \Rightarrow \begin{bmatrix} (-e^{-t}|_0^T) & 0 \\ (-e^{-t} + \frac{1}{2}e^{-2t}|_0^T) & (\frac{-1}{2}e^{-2t}|_0^T) \end{bmatrix} B$$

$$= \begin{bmatrix} 1/2 & 0 \\ 1/8 & 3/8 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 11/8 \end{bmatrix}$$

Given the following discrete time system model:

$$\mathbf{x}(n+1) = G\mathbf{x}(n) + Hu(n) = \begin{bmatrix} 0 & 1 \\ \frac{3}{8} & \frac{5}{4} \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(n), \tag{5}$$

$$y(n) = C\mathbf{x}(n) = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

[4 pts] b. Find transfer function for the system in eqn.(5) with input $U(z)$ and output $Y(z)$:

$$\boxed{M(z) = \frac{Y(z)}{U(z)} = \frac{1}{(z - 3/2)(z + 1/4)}}$$

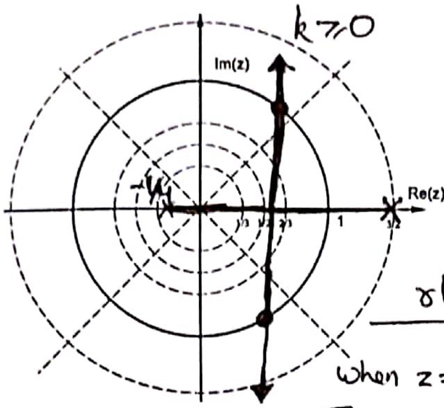
$$C(zI - G)^{-1}H = [1 \quad 0] \begin{bmatrix} z & -1 \\ -3/8 & z - 5/4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= [1 \quad 0] \left(\frac{1}{z^2 - 5/4 z - 3/8}\right) \begin{bmatrix} z - 5/4 & 1 \\ 3/8 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{z^2 - 5/4 z - 3/8}$$

$$= \frac{1}{(z - 3/2)(z + 1/4)}$$

Problem 6, continued

[3 pts] c. Let control input $u(n) = K(r(n) - y(n))$ where K is output feedback gain, and $r(n)$ is the input, be applied to the system above (eqn. 5). Plot the root locus for $M(z)$ with unity gain feedback:



$M(z) = \frac{1}{(z-3/2)(z+1/4)}$

for $k<0$ $\quad -1/4 \quad 3/2$

breakaway @ $\frac{(-1/4 + 3/2)}{2} = 5/8$

rlocus intersects unit circle when $\rightarrow z=1, z=\frac{5}{8} \pm \frac{\sqrt{39}}{8}j$

$1 = \sqrt{(5/8)^2 + y^2}$

$\Rightarrow \frac{64-25}{64} = y^2 \Rightarrow y = \frac{\sqrt{39}}{8}$

poles: $z^2 - 5/4 z + (k-3/8) = 0$

when $z=1$: $1^2 - 5/4 + k - 3/8 \Rightarrow \boxed{k = 5/8}$

$C(zI - G_{CL})^{-1}H = \frac{1}{z^2 - 5/4 z + (k-3/8)}$

when $z = \frac{5}{8} \pm \frac{\sqrt{39}}{8}j$: $\frac{1}{32}(\cancel{7} + \cancel{5j\sqrt{39}}) - \frac{1}{32}(25 + \cancel{5\sqrt{39}j}) + k - \frac{3}{8} = 0$

$\Rightarrow \boxed{k = 11/8}$

[3 pts] d. The system in eqn. (5), with input r, has control input $u(n) = K(r(n) - y(n))$ applied, where K is output feedback gain. Find range of K for the system to be stable.

$\underline{5/8 < K < 11/8}$

$x(n+1) = G(x(n)) + HKr(n) - HKCx(n) = (G - HKC)x(n) + HKr(n)$

$G - HKC = \begin{bmatrix} 0 & 1 \\ 3/8 & 5/4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} K \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3/8 - K & 5/4 \end{bmatrix}$

$[zI - (G - HKC)]^{-1} = \begin{bmatrix} z & -1 \\ k-3/8 & z-5/4 \end{bmatrix}^{-1} = \frac{1}{z^2 - \frac{5}{4}z + (k-3/8)} \begin{bmatrix} z-5/4 & 1 \\ 3/8-k & z \end{bmatrix}$

[2 pts] e. Find the final value of $x(n)$ for a step input to the open loop system in eqn.(5), that is $\lim_{n\to\infty} x(n)$ for a unit step input.

Checking stability: $eig(G) = +3/2, -1/4$. Here, $3/2 > 1 \Rightarrow$ system is not BIBO stable.

$\Rightarrow$ step input might give unbounded output

OR

$x_1(n+1) = x_2(n)$

$x_2(n+1) = \frac{3}{8}x_1(n) + \frac{5}{4}x_2(n) + 1$

Assuming $x_i(0) = 0$, x_2 will always increase by at least $1 + \frac{1}{4}x_2(n)$. Also, $x_1 \& x_2 > 0 \Rightarrow \boxed{x_2 \to \infty}$ and, x_1 follows $x_2 \Rightarrow \boxed{x_1 \to \infty}$

[4 pts] f. A linear time invariant causal discrete time system with input $u(k)$ and output $x(k)$ has z transform

$$X(z) = \frac{1}{(1 - \frac{3}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}$$

The system is driven by a discrete unit step, $u(k) = 1$ for $k \geq 0$. Find the output $x(k)$ for $k \geq 0$.

$X(z) = \frac{z}{(z-1)\left(\frac{z-3/4}{z}\right)\left(\frac{z-1/2}{z}\right)} = \frac{z^3}{(z-1)(z-3/4)(z-1/2)}$

$x(k) = \underline{8 - 9\left(\frac{3}{4}\right)^k + 2\left(\frac{1}{2}\right)^k}$

$\frac{X(z)}{z} = \frac{z^2}{(z-1)(z-3/4)(z-1/2)} = \frac{A}{z-1} + \frac{B}{z-3/4} + \frac{C}{z-1/2}$

$X(z) = \frac{8z}{z-1} + \frac{-9z}{z-3/4} + \frac{2z}{z-1/2}$

$z^2 = A(z-3/4)(z-1/2) + B(z-1)(z-1/2) + C(z-1)(z-3/4)$

$z=1 \Rightarrow 8 = A$

$z = 3/4 \Rightarrow 9/16 = -1/16 B \Rightarrow B = -9$

$z = 1/2 \Rightarrow 1/4 = 1/8 C \Rightarrow C = 2$