1. Root Loci

\[ G(s) = \frac{(s+10)(s+20)}{(s+30)(s^2 - 20s + 200)} \]

\[ G'(s) = \frac{G(s)}{s+1} \]

\[ T(s) = \frac{C(s)}{R(s)} \]

\[ T(s) = \frac{kG(s)}{1 + kG(s)} \]

A) Root Loci

Solving for Poles/Zeros of Gain H∞

\[ G(s) = \frac{(s+10)(s+20)}{(s+30)(s^2 - 20s + 200)} \]

- Poles: \( s = -30, 10 \pm 10j \) \( (n = 3) \)
- Zeros: \( s = -10, -20 \) (and one infinite zero) \( (m = 2) \)

(1) Note: Loci begins at CL pole and ends at CL zero.
(2) Loci is symmetric about the real axis.
(3) Loci exists (for \( k > 0 \), on real axis) to the left of odd numbers of finite poles and zeros.
(4) Loci branches equal to number of CL poles \( (n_{clp} = 3) \)
(5) Behavior at Infinity:

\[ \theta_n = -(2L+1)\pi / (n-m) \cdot \frac{(-2L+1)\pi}{1} = -(2L+1)\pi \]

\[ \Rightarrow \theta_n = \pm \pi, \pm 3\pi, \ldots \]

Inceptor: \((\text{zeros} - \text{poles}) / (n-m) = (-30-20+10+20) / 1 = -20.\)

(6) Break Away / Break In Locations:

\[ \frac{2}{\sigma + \sigma_j} = \frac{2}{\sigma - \sigma_i} \]

\[ \L (\sigma + 10) + L (\sigma + 20) = \frac{1}{(\sigma + 20) - \frac{1}{(\sigma + 10 + 10j)} - \frac{1}{(\sigma - 10 - 10j)} \]

\[ \Rightarrow \sigma_1 = -14.97 \quad \text{Only care about real locations} \]

\[ \sigma_2 = 12.498 \]

\[ \sigma_3 = -28.81 + 22.97j \]

(8) Angle of Departure from Complex Pairs:

\[ \tan^{-1} \left( \frac{\Im(\theta)}{\Re(\theta)} \right) = \tan^{-1} \left( \frac{-2L+1}{\pi} \right) \]

\[ \theta_e = \tan^{-1} \left( \frac{-2L+1}{\pi} \right) \]

\[ \Rightarrow \theta_e = 2.111 \text{ rad} = 121^\circ \quad (\text{for pole } 10 + 10j) \]

(7) \(j\omega\)-axis Crossing via Routh–Hurwitz

\[ T_{10} = \frac{(s+10)(s+20)}{(s^3 - 20s^2 + 200s + 400)(s+10)(s+20)} \]

\[ = \frac{(s^3 - 30s^2 + 200)}{(s^3 + (10 + k)s^2 + (400 + 30k)s + (6000 + 200k))} \]

\[ \begin{array}{c|ccc}
\tau & 1 & -400 - 30k & 0 \\
\tau^2 & 1 & 400 + 200k & 0 \\
\tau^3 & 10 + k & 600 + 200k & 0 \\
\tau^4 & b_1 & 0 & 0 \\
\tau^5 & c_1 & 0 & 0 \\
\tau^6 & d_1 & 0 & 0 \\
\tau^7 & e_1 & 0 & 0 \\
\tau^8 & f_1 & 0 & 0 \\
\tau^9 & g_1 & 0 & 0 \\
\tau^{10} & h_1 & 0 & 0 \\
\tau^{11} & i_1 & 0 & 0 \\
\tau^{12} & j_1 & 0 & 0 \\
\tau^{13} & k_1 & 0 & 0 \\
\tau^{14} & l_1 & 0 & 0 \\
\tau^{15} & m_1 & 0 & 0 \\
\tau^{16} & n_1 & 0 & 0 \\
\tau^{17} & o_1 & 0 & 0 \\
\tau^{18} & p_1 & 0 & 0 \\
\tau^{19} & q_1 & 0 & 0 \\
\tau^{20} & r_1 & 0 & 0 \\
\tau^{21} & s_1 & 0 & 0 \\
\tau^{22} & t_1 & 0 & 0 \\
\tau^{23} & u_1 & 0 & 0 \\
\tau^{24} & v_1 & 0 & 0 \\
\tau^{25} & w_1 & 0 & 0 \\
\tau^{26} & x_1 & 0 & 0 \\
\tau^{27} & y_1 & 0 & 0 \\
\tau^{28} & z_1 & 0 & 0 \\
\end{array} \]

\[ b_1 = -400 + 30k - (6000 + 200k) / (10 + k) \]

\[ k = 23.93 \quad (5 + 5/2 \sqrt{121}) \]

\[ c_1 = 600 + 200k \]

\[ \tau = 30 \]
At $\zeta = 23.93$:

$$T(s) = \frac{(s+10)(s+20)}{(s^3 + 38.93s^2 + 317.9s + 10986)}$$

$$s_1 = 17.83j; \quad s_2 = -17.83j;$$
$$s_3 = -33.93$$

B) Stability of System

From the above Routh-Hurwitz Table:

$$k \in [23.93, \infty) \quad \text{gives stability}$$

C) Damping $\zeta = 0.707$ for 2nd-Order Approx.

From Root Locus on the first page, we see that $\zeta = 0.707$ corresponds to $\theta = 45^\circ$ in the complex plane. We must find $K$ for the dominant CL Poles to exist along this $\theta = 45^\circ$ line.

$$\|\text{G}(s)H(s)\| = 1/K$$
$$z_\theta = (2k+1)\pi$$

Pole

$$- \tan^{-1}\left(\frac{r'}{-10-r'}\right) - \tan^{-1}\left(\frac{r'+10}{-10-r'}\right) + \tan^{-1}\left(\frac{r'\varepsilon}{r'-10}\right) + \tan^{-1}\left(\frac{r'/r'-20}{r'-30}\right) = (2k+1)\pi$$

$$r' = 13.04 \quad \varepsilon = 1$$

$$((10+r')^2 - (10-r')^2)^{1/2} = (10+r')^2 + (10-r')^2)^{1/2}$$

$$((r')^2 + (r'-10)^2)^{1/2} = ((r')^2 + (r'-20)^2)^{1/2}$$

$$\varepsilon = \frac{1}{K}$$

$$K = 81.85$$
D) Damping \( \zeta = 1.00 \) (Critical Damping)

\[ \zeta = 1 \]
\[ \downarrow \]
\[ \Theta = 0 \quad (\zeta = \cos(\Theta)) \]

so we want \( K \) at the break-in point of \( \sigma = -14.97 \).

\[ |G_{po}H_{po}| = \frac{1}{k} \]
\[ \left( (10)^2 + (10 + 14.97)^2 \right)^{-1/2} \cdot \frac{1}{(10)^2 + (10 + 14.97)^2} \cdot (10 + 14.97) \cdot (1 - 10 + 14.97) \cdot \frac{1}{1 - 20 + 14.97} \cdot (1 - 30 + 14.97) \cdot \frac{1}{1 - 40 + 14.97} = \frac{1}{k} \]

\[ K = 434.9 \]

E) See Attached Figure.

Neither system fully reproduces the desired response, due to the third pole (and the zeros). The \( \zeta = 1 \) critically damped response is not actually critically damped and has overshoot (41.7%) while the \( \zeta = 0.707 \) underdamped response shows greater overshoot (65.7%) than expected from \( 90.5 \% + 100 \cdot e^{-\frac{\pi}{\sqrt{1-0.707}}} = 4.35 \% \).
2. a) \( G(s) = \frac{s^2 + 19s - 20}{s^2 - 10s + 10\epsilon} \)

\( \zeta(s) = \frac{G_{ee}(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \)

\( = \frac{s^2 + 19s - 20}{s^2 + 19s - 20 + s^2 - 10s + 10\epsilon} \)

\( \Delta(s) = 2s + 9s - 20 + 10\epsilon \)

b) Rearranging to put in form:

\( \frac{N(s)}{1 + cT(s)} \)

\( \Delta(s) = 2s^2 + 9s - 20(1 + c \frac{10}{2s^2 + 9s - 20}) \)

\( T(s) = \frac{10}{2s^2 + 9s - 20} \)

\( N(s) = \frac{s^2 + 19s - 20}{2s^2 + 9s - 20} \)

Poles: \( \{1.63, -6.13\} \)

Zeros: None

\( \sigma_0 = -2.25 \)
3. a) Finding \( \frac{\Theta_{b0}(s)}{\Theta_{c0}(s)} \):

\[
\Theta_{b0} = \frac{1}{s^2} \left( k_2(\Theta_{cb} - \Theta_{b0}) + sk_1\Theta_{cb} - s\Theta_{db}k_3 \right)
\]

\[
\Theta_{db} = k_2(\Theta_{cb} - \Theta_{b0}) + sk_1\Theta_{cb} - s\Theta_{db}k_3
\]

\[
\frac{\Theta_{db}}{\Theta_{cb}} = \frac{k_2 + k_1s}{s^2 + k_2s + k_2}
\]

Putting into form:

\[
\frac{N(s)}{1 + k_2T_2(s)}
\]

During numerator + denominator \( s = \Theta_{db} \) by & + \( s^2 + k_2 \)

\[
N(s) = \frac{k_2 + k_1s}{s^2 + k_2}
\]

\[
T_2(s) = \frac{s}{s^2 + k_2}
\]

zeros: 0, poles: 0 \( \pm \sqrt{k_2} \) j

Departure angle: 0

\( \Theta_a = -M \)

Break-in: \( \sigma_a = \frac{1}{\sigma_0 + k_3} \)

\( \sigma_0^2 + k_2 = 2\sigma_0^2 \)

b) CL "zero" at 0, -\( \infty \)
3. c) massaging $\frac{\theta_{do}}{\delta_{wo}}$ into form $\frac{N_2(s)}{1+K_2 T_2(s)}$

$$T_2(s) = \frac{1}{s^2 + K_3 s}$$

$$N_2(s) = \frac{K_1 + K_3 s}{s^2 + K_3 s}$$

**closed-loop zero** at $\frac{-K_3}{2} + j\infty$
4. Root Locus

\[ G(s) = \frac{s^2 + 3s + 2}{s^2 - 4s + 5} \]
\[ G'(s) = KG(s) \quad \text{(with } G'(s) \text{ in unity feedback)} \]
\[ T(s) = \frac{G(s)}{1 + KG(s)} = \frac{K(s^2 + 3s + 2)}{(s^2 - 4s + 5) + KH^2(s^2 + 3s + 2)} \]

A) RL Rules:

- **Poles (of G(s))**: \[ s = 2 \pm j \]
- **Zeros (of G(s))**: \[ s = -2, 1 \]

Poles and zeros are equal, so no behavior at infinity.

- **Real Axis Segments**:
  - For \( k > 0 \): \([-2, -1]\)
  - For \( k < 0 \): \((-\infty, -2), (-1, \infty)\)

- **Break-away / Break-in Points**:
  - For \( k > 0 \):
    \[ \frac{s^2}{\sigma + 2} = \frac{1}{\sigma + p_1} \]
  - For \( k < 0 \):
    \[ \frac{1}{\sigma - (a)} + \frac{1}{\sigma - (b)} = \frac{1}{\sigma - (2+j)} + \frac{1}{\sigma - (2-j)} \]

\[ \sigma = \frac{3}{2} \pm \sqrt{\frac{17}{4}} \]

\[ \sigma = 1.434, \ 2.291 \]

- For \( k > 0 \):
  \[ \theta_0 = \frac{\pi}{2} (2l + 1) \pi \]
  \[ -\theta_e - \tan^{-1}(1/2) + \tan^{-1}(1/4) = (2l + 1) \pi \]
  \[ \theta_0 = 2.138 \text{ rad} \]

- For \( k < 0 \): \[ \frac{5}{4} \theta_0 = (2l) \pi \]

Angle of Departure from Complex Poles:

\[ \theta_e = \frac{\pi}{2} + \tan^{-1}(1/2) + \tan^{-1}(1/4) = (2l + 1) \pi \]

\[ \theta_e = \frac{\pi}{2} \text{ rad} \]

\[ (k = 0) \]

\[ \theta_e = -1.004 \text{ rad} \]

\[ (\text{rad from } \theta_e \text{ for } k > 0) \]
B/D) jw-crossing via Routh Hurwitz:

\[ (1 + kg) = 5^2 (1 + k) + 5 (-4 + 3k) + (5 + 2k) \]

| \( S^2 \) | \( 1 + 4k \) | \( 5 + 2k \) | 0 |
| \( S^1 \) | \( -4 + 3k \) | 0 | 0 |
| \( S^0 \) | \( 5 + 2k \) | 0 | 0 |

All Positive: \( k > -1 \) (3)
\[ k > 4/3 \] (3')
\[ k > -5/2 \] (4)

All Negative: \( k < -1 \) (3)
\[ k < 4/3 \] (3')
\[ k < -5/2 \] (4')

The system is stable for \( k < (-\infty, -5/2] \) and \([4/3, \infty)\)

To find jw-crossings, evaluate TF at these gain values:

At \( K = 4/3 \):
\[ \text{Tr} = (4/3) (5^2 + 3\times 2) \]
\[ = 152.5 \]
\[ \text{Det} = (5^2 - 4\times 5) + (4/3)(5^2 + 3\times 2) \]
\[ = 181.3 \]
\[ s_1 = -j(23/7) \]
\[ s_2 = 1.813 \]

At \( K = -5/2 \):
\[ s_1 = 0 \]
\[ s_2 = -23/3 \]
Note: Since we have a system with 2 poles & 2 zeros, at $k = -1$ we will have a pole cancellation in the closed loop transfer function ($T(s)$) which is why one pole moves $\sigma \to \infty$ (for $k = -1^+$) and $\sigma \to -\infty$ (for $k = -1^-$).
5. a) apply design criteria to dominant poles under 2nd order approx:

\[ 0.05 = 20 \Rightarrow \delta = 45.6 \Rightarrow \theta = 62.87 \]

occurs \( K = 1.160 \)

poles @ \( K = 1.160 \) \( \Rightarrow \)

poles = \[
-3.195 \pm 6.239
\]

\[-23.610\]

3rd pole more than 5x further than real component of 1st 2 poles \( \Rightarrow \) 2nd order approximation is "valid"

b) \( \delta^d = 45.6 \), 4x increase in \( 3 \) \( \omega_n \) must come from \( \omega_n \)

\[
\omega_n^d = 4 \omega_n, \sim \omega_n \text{ from part (a)}
\]

\[
\Rightarrow \omega_{n,d} = \frac{3.195}{45.6} = 0.07
\]

\( \omega_n^d = 28 \)

\( \sigma^d = \delta^d \omega_n^d = 12.768 \)

dominant poles must be at \( -12.77 \pm 24.919 \)j

phase of 3 existing poles @ \( -12.77 \pm 24.919 \)j

\[
\phi = \tan^{-1} \left( \frac{24.92}{12.77} \right) + \tan^{-1} \left( \frac{24.92}{-2.77} \right) + \tan^{-1} \left( \frac{24.92}{7.33} \right)
\]

\[
= (117.13 + 96.54 + 73.84)
\]
5) cont'd

\[ \theta = 267.3^\circ \]

Adding zero should bring \( \theta \) to \(-180^\circ \)

\[ \theta + \angle(zero) = -180^\circ \]

\[ \angle(zero) = 107.3^\circ \]

Zero should be on real axis \( \Rightarrow \tan^{-1}\left( \frac{24.919}{-12.71.02} \right) = 107.3^\circ \)

\[ \Rightarrow \sigma_z = -5.0038 \]

Asymptote: \(-12.49 \)

\[ \sigma_z = -14.47 \]

Checking gain @ limits of design criteria

\[ K_p = 3,960 \text{ via MATLAB} \]

\[ \Rightarrow K_a = 757.43 \]

Checking CL poles under these gains:

Poles: \([-12.75 \pm 25.96j, -4.49 \approx \text{close to zero, approx cancels, 2nd order poles}\]

C.) in (a)(b) for hand, MATLAB attached dominant

D.) attached. Note PD \%OS < 20\%, P2 cancellation not perfect

E.) (a) type 1 sys \( \Theta_s = 0 \)

(b) type 1 sys \( \Theta_s = 0 \)
Uncompensated RL

PD Compensated RL

Step Comparison