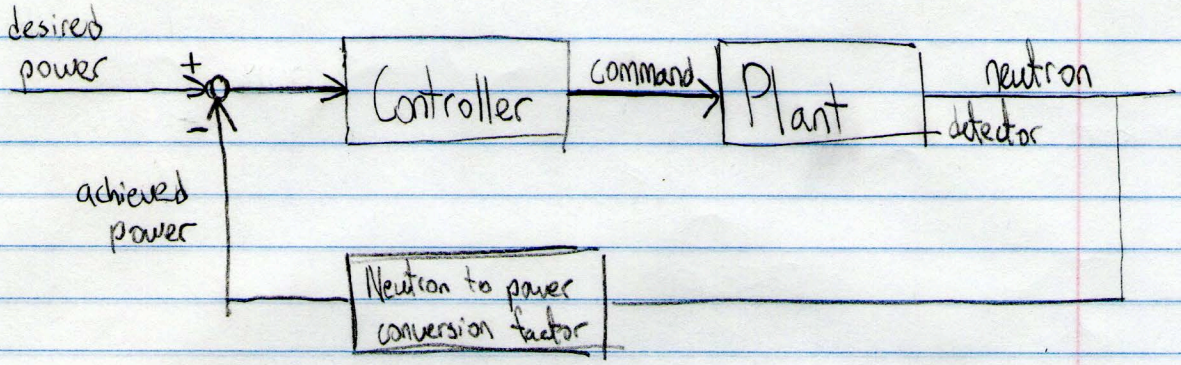
EE128/ME134

Problem Set 1 Solutions

1.



2. $$\frac{1}{(s+1)^2(s+4)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+4} + \frac{D}{(s+4)^2} \quad (1)$$

isolating each term, then letting s take values to reduce:

B: multiply by $(s+1)^2$ each side

multiplying both sides of (1) by $(s+1)^2$, let $s \to -1$

$$B = \frac{1}{(s+4)^2} - (s+1)A - (s+1)^2\frac{C}{s+4} - (s+1)^2\frac{D}{(s+4)^2} = 1/9 \quad (2)$$

D: multiply (1) by $(s+4)^2$, $s \to -4$

$$D = \frac{1}{(s+1)^2} - \frac{(s+4)^2 A}{s+1} - \frac{B}{(s+1)^2}(s+4)^2 + (s+4)C \quad (3)$$

A: differentiate (2) w.r.t s

$$\frac{-2}{(s+4)^3} = A + \frac{2(s+4)C(s+1) - (s+1)^2C}{(s+4)^2} + \frac{2(s+1)(s+4)^2D - 2D(s+1)^2(s+4)}{(s+4)^4}$$

let $s \to -1$ to reduce terms:

$$A = \frac{-2}{27}$$

C: differentiate (3) w.r.t. s

$$\frac{2}{(s+1)^3} = C + (s+4)\left(\frac{d}{dt}\frac{A(s+4)}{s+1} - \frac{B(s+4)}{(s+1)^2}\right)$$

let $s \to -4$

$$C = 2/27$$

PS1 cont'd

2. (cont'd)

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2(s+4)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{-2}{27(s+1)} + \frac{1}{9(s+1)^2} + \frac{2}{27(s+4)} + \frac{1}{9(s+4)^2}\right\}$$

taking inverse laplace transform of each individually:

$$\mathcal{L}^{-1}\{\dots\} = -2/27\, e^{-t} + 1/9\, t e^{-t} + 2/27\, e^{-4t} + 1/9\, t e^{-4t}$$

3. i) $\dfrac{1}{s^2+8s+16} = \dfrac{1}{(s+4)^2}$

$$h_1(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\} = \boxed{t e^{-4t} u(t)}$$

ii) $\dfrac{s}{s^2+8s+16} = \dfrac{s}{(s+4)^2}$

$$= s H_1(s)$$

$$\mathcal{L}^{-1}\{s H_1(s)\} = \frac{d}{dt} h_1(t)$$

$$h_2(t) = \frac{d}{dt}\left(t e^{-4t} u(t)\right)$$

$$= \left(e^{-4t} - 4t e^{-4t}\right) u(t) + \underbrace{t e^{-4t}}_{0}\, \delta(t)$$

$\delta(t) = 0 \ \forall t \neq 0$

@ $t = 0$, $(t e^{-4t}) = 0$

$$h_2(t) = (1-4t) e^{-4t} u(t)$$

iii) $\mathcal{L}^{-1}\left\{\dfrac{s+3}{(s+4)^2}\right\} = \mathcal{L}^{-1}\left\{\dfrac{s}{(s+4)^2} + \dfrac{3}{(s+4)^2}\right\}$

$$= \mathcal{L}^{-1}\{H_2(s) + 3H_1(s)\}$$

$$h_3(t) = h_2(t) + 3h_1(t)$$

$$h_3(t) = \left((1-4t) e^{-4t} + 3t e^{-4t}\right) u(t) + \underbrace{t e^{-4t}}_{0}\, \delta(t)$$

$$= (1-t) e^{-4t} u(t)$$

PS1 Cont'd

3 cont'd

iv) improper fraction, first doing polynomial long div

$$\begin{array}{r} 1 \\ s^2+8s+16 \overline{\big) s^2 } \\ -\ s^2+8s+16 \\ \hline -8s-16 \end{array} \Bigg\} \rightarrow H_4(s) = 1 + \frac{-8s-16}{(s^2+8s+16)}$$

$$= 1 - 8\frac{s}{(s+4)^2} - 16\frac{1}{(s+4)^2}$$

$$\mathcal{L}^{-1}\{H_4(s)\} = \mathcal{L}^{-1}\{1 - 8H_2(s) - 16H_1(s)\}$$
$$= \delta(t) - 8e^{-4t}u(t) + 32te^{-4t}u(t) - 16te^{-4t}u(t)$$
$$= \delta(t) - 8e^{-4t}u(t) + 16te^{-4t}u(t)$$

v) again, improper

$$\begin{array}{r} 1 \\ s^2+8s+16 \overline{\big) s^2 + 4} \\ -(s^2+8s+16) \\ -8s-12 \end{array} \Bigg\} \rightarrow H_5(s) = 1 + \frac{-8s-12}{s^2+8s+16}$$

$$\mathcal{L}^{-1}\{H_5(s)\} = \delta(t) - (8e^{-4t} + 16te^{-4t})u(t)$$

4. $y_i(0^+) = \lim_{s\to\infty} sY_i(s)$ $\qquad$ $\lim_{t\to\infty} y_i(t) = \lim_{s\to 0} sY_i(s)$

Initial Value Theorem $\qquad$ Final Value Theorem

using these two theorems; and l'hopital's rule

i) $sY_1(s) = s^2/s+4$ $\qquad$ $\lim_{s\to 0} sY_1(s) = 0$

$Y_1(s)$ is not strictly proper, can't use IVT

$y_1(t) = \mathcal{L}^{-1}\{Y_1(s)\} = \mathcal{L}^{-1}\{1 + -4/s+4\} = \delta(t) - 4e^{-4t}$

$\lim_{t\to 0^+} y_1(t)$ does not exist $\rightarrow$ $\delta(0)$ undefined

ii) $sY_2(s) = \frac{(s+1)^3}{s^2}$

IVT: $Y_2(s)$ improper, again $y_2(0^+)$ is undefined

FVT: two poles at $s=0$, $s\to 0$ outside ROC but $y_2(t)\to\infty$ as $t\to\infty$

4 cont'd

iii) $sY_3(s) = \frac{s(s+2)^3}{(s+1)^3}$

not strictly proper

$y_3(0^+)$ undefined

$\lim_{s\to 0} sY_4(s) = 0$

iv) $sY_4(s) = \frac{s^3+s}{s+4}$

not strictly proper

$y_4(0^+)$ undefined

$\lim_{s\to 0} sY_4(s) = 0$

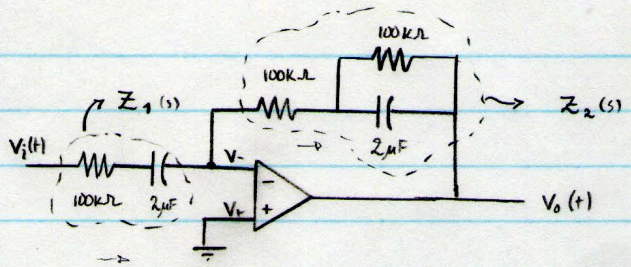
v) $sY_5(s) = \frac{s}{(s+1)(s+7s+12)s}$

$\lim_{s\to \infty} sY_5(s) = 0$

$\lim_{s\to 0} sY_5(s) = 1/12$

	IV	FV
i	undefined	0
ii	undefined	∞
iii	undefined	0
iv	undefined	0
v	0	1/12

5.



$V_+ = V_- = 0V \quad \Leftrightarrow \quad V_- = V_1(t)$

$V_i(s) / Z_{1(s)} = - V_o(s) / Z_2(s) \qquad (V_i - 0)/Z_i = i_{in} = i_{out} = (0 - V_o)/Z_2$

$\hookrightarrow Z_{1(s)} = R + 1/Cs \qquad$ with $R = 100\ k\Omega$, $C = 2\ \mu F$

$Z_{1(s)} = (RCs + 1) / (Cs)$

$\hookrightarrow Z_{2(s)} = R + (1/R + Cs)^{-1}$

$= R + 1/(Cs + 1/R)$

$Z_{2(s)} = (RCs + 2) / (Cs + 1/R)$

$H_{(s)} = V_{o(s)} / V_{i(s)}$

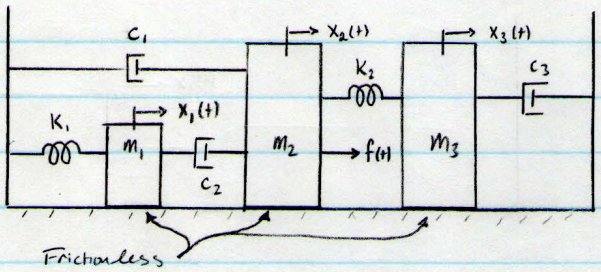
$= - Z_{2(s)} / Z_{1(s)}$

$= (RCs + 2)(Cs) / (RCs + 1)(Cs + 1/R)$

$$H_{(s)} = (s)((RC^2)s + 2C) / ((RC^2)s^2 + (2C)s + 1/R)$$

$$H_{(s)} = (s)((4\times10^{-7})s + 4\times10^{-6}) / ((4\times10^{-7})s^2 + (4\times10^{-6})s + 1\times10^{-5})$$

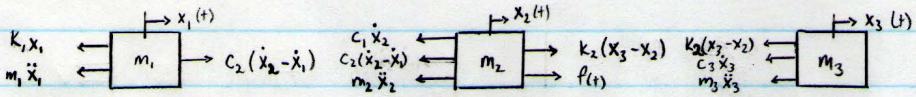
6.



$m_1 = m_2 = m_3 = 4\,kg$

$K_1 = K_2 = 6\,N/m$

$c_1 = c_2 = c_3 = 2\,Ns/m$



$$\sum F_1(t) = m_1\ddot{x}_1 \Rightarrow \quad m_1\ddot{x}_1 + K_1x_1 = c_2(\dot{x}_2 - \dot{x}_1) \tag{1}$$

$$\sum F_2(t) = m_2\ddot{x}_2 \Rightarrow \quad m_2\ddot{x}_2 + c_1\dot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) = K_2(x_3 - x_2) + f(t) \tag{2}$$

$$\sum F_3(t) = m_3\ddot{x}_3 \Rightarrow \quad m_3\ddot{x}_3 + c_3\dot{x}_3 + k_2(x_3 - x_2) = 0 \tag{3}$$

$$m_2 s^2 X_1(s) + K_1 X_1(s) = c_2 s X_2(s) - c_2 s X_1(s)$$

$$X_1(s)\,(m_2 s^2 + c_2 s + K_1) = X_2(s)\,(c_2 s) \tag{4}$$

$$m_2 s^2 X_2(s) + c_1 s X_2(s) + c_2 s X_2(s) - c_2 s X_1(s) = K_2 X_3(s) - K_2 X_2(s) + F(s)$$

$$X_2(s)\,(m_2 s^2 + (c_1 + c_2)s + K_2) + X_1(s)\,(-c_2 s) + X_3(s)\,(-K_2) = F(s) \tag{5}$$

$$m_3 s^2 X_3(s) + c_3 s X_3(s) + K_2 X_3(s) - K_2 X_2(s) = 0$$

$$X_3(s)\,(m_3 s^2 + c_3 s + K_2) = X_2(s)\,(K_2) \tag{6}$$

(4) & (6) into (5) ...

$$X_2(s)\left(m_2 s^2 + (c_1 + c_2)s + K_2\right) + X_2(s)\left(-c_2^2 s^2 / (m_2 s^2 + c_2 s + K_1)\right) + X_2\left(-K_2^2 / (m_3 s^2 + c_3 s + K_2)\right) = F(s)$$

given $m_1 = m_2 = m_3 = 4\,kg$ / $K_1 = K_2 = 6\,N/m$ / $c_1 = c_2 = c_3 = 2\,Ns/m$.

$$X_2(s)\left((ms^2 + 2cs + K) + (-c^2 s^2 - K^2)/(ms^2 + cs + K)\right) = F(s)$$

$$X_2(s)/F(s) = (ms^2 + cs + K)/((ms^2 + 2cs + K)(ms^2 + cs + K) - c^2 s^2 - K^2)$$

$$\Rightarrow \boxed{sX_2(s)/F(s) = (s)(ms^2 + cs + K)/((ms^2 + 2cs + K)(ms^2 + cs + K) - c^2 s^2 - K^2)} \tag{7}$$

$$\boxed{sX_2(s)/F(s) = (s)(4s^2 + 2s + 6)/((4s^2 + 4s + 6)(4s^2 + 2s + 6) - 4s^2 - 36)}$$

7. Electrical Circuit Equivalences: (Using Series Analog)

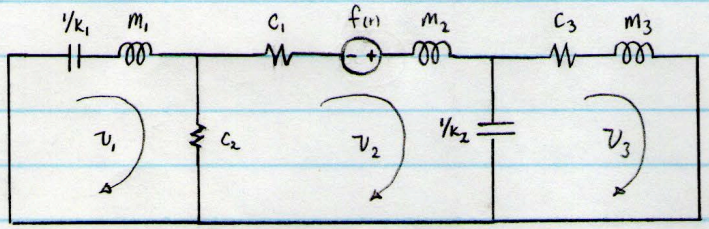
↳ Mass (m (kg)) → Inductor (m (H)) $L = m$

↳ Damper (c (Ns/m)) → Resistor (c (Ω)) $R = c$

↳ Spring (k (N/m)) → Capacitor ($1/k$ (F)) $C = 1/k$

↳ Applied Force ($f(t)$) → Voltage Source ($f(t)$ (V))

↳ Velocity ($\dot{x}(t) = v(t)$) → Mesh Current ($v(t)$ (A))



$V_R(s) = V(s) R$ (c)

$V_C(s) = V(s)(1/Cs)$ (1/k)

$V_L(s) = V(s)(Ls)$ (m)

↳ Current

② KVL: $F(s) = (m_2 s + K_2/s + c_2 + c_1) V_2(s) + (-K_2/s) V_3(s) + (-c_2) V_1(s)$ (1)

① KVL: $0 = (m_1 s + c_2 + K_1/s) V_1(s) + (-c_2) V_2(s)$ (2)

③ KVL: $0 = (m_3 s + K_2/s + c_3) V_3(s) + (-K_2/s) V_2(s)$ (3)

(2) & (3) into (1) ...

$$F(s) = (m_2 s + K_2/s + c_2 + c_1) V_2(s) + (-c_2^2 / (m_1 s + c_2 + K_1/s)) V_2(s) + ((-K_2^2/s^2)/(m_3 s + K_2/s + c_3)) V_2(s)$$

given $m_1 = m_2 = m_3 = 4$ H / $1/k_1 = 1/k_2 = 1/6$ F / $c_1 = c_2 = c_3 = 2\ \Omega$

$$F(s) = \left((m s + 2c + k/s) + (-c^2 - K^2/s^2)/(ms + c + K/s) \right) V_2(s)$$

$$V_2(s) / F(s) = (ms + c + K/s) / ((ms + 2c + K/s)(ms + c + K/s) - c^2 - K^2/s^2)$$

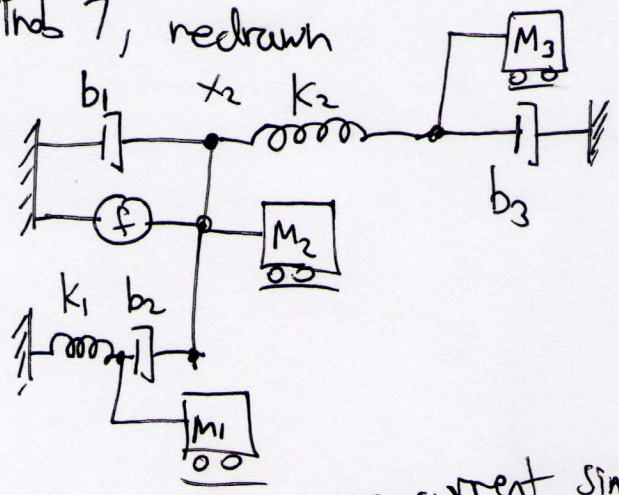
$$\Rightarrow \boxed{V_2(s) / F(s) = (s)(ms^2 + cs + K) / ((ms^2 + 2cs + K)(ms^2 + cs + K) - c^2 s^2 - K^2)}$$

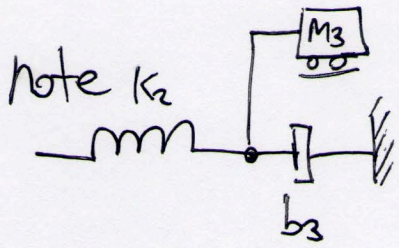
with $s X_2(s) = V_2(s)$

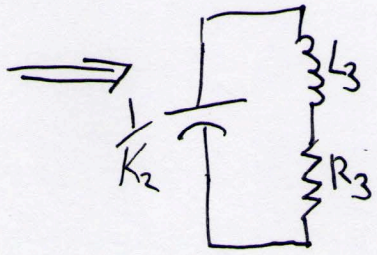
Same as (7) in Problem 6.

PS1

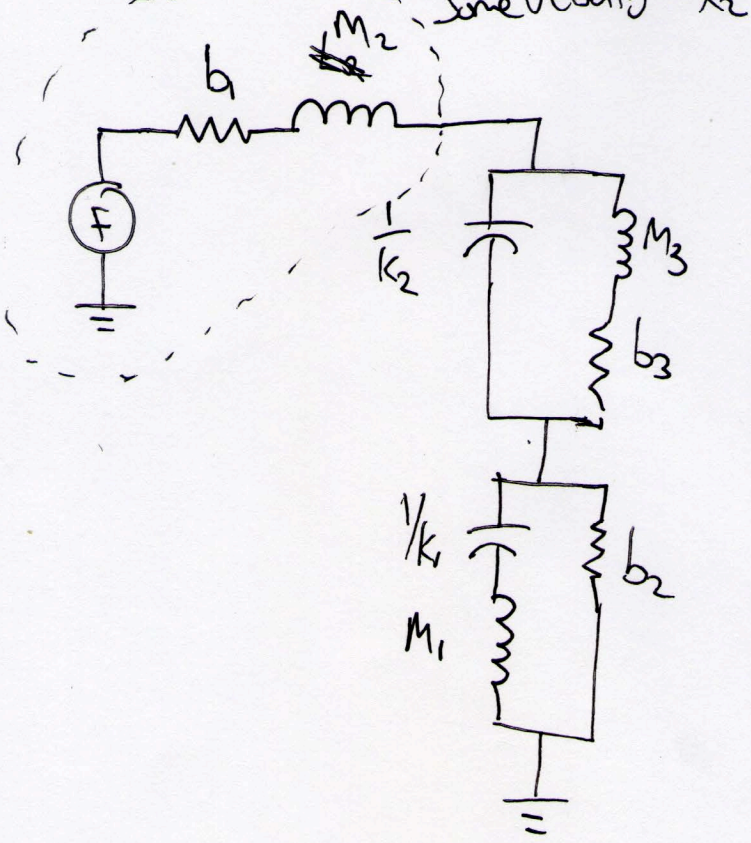
Prob 7, redrawn

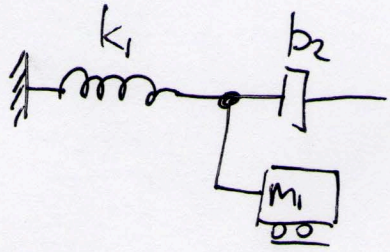


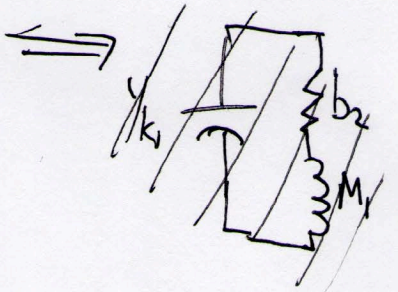


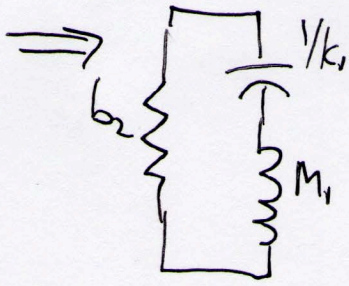


same current since same velocity = $\dot{x}_2$









This circuit is identical to previous page. It shows more clearly the groups of elements which have same force/velocity [voltage/current].