Stability for LTI systems, [1, p. 302]

Total response of a system

\[ c(t) = c_{\text{forced}}(t) + c_{\text{natural}}(t) \]

Stability for LTI systems

- Natural response as \( t \rightarrow \infty \)
  - Stable: \( \rightarrow 0 \)
  - Unstable: Grows without bound
  - Marginally stable: Neither decays nor grows but remains constant
- Total response (BIBO)
  - Stable: Every bounded input yields a bounded output
  - Unstable: Any bounded input yields an unbounded output
    - Marginally stable: Some bounded inputs yield unbounded outputs
- Stability \( \implies \) only the forced response remains

Stability for LTI systems in terms of pole locations

- Closed-loop TF poles
  - Stable: Only in LHP
  - Unstable: At least 1 in RHP and/or multiplicity greater than 1 on the imaginary axis
  - Marginally stable: Only imaginary axis poles of multiplicity 1 and poles in the LHP
6 Stability

6.1 Introduction

6.2 Routh-Hurwitz criterion

6.3 Routh-Hurwitz criterion: special cases

6.4 Routh-Hurwitz criterion: additional examples

6.5 Stability in state space

History interlude

Edward John Routh

- 1831 – 1907
- English mathematician
- 1876 – Proposed what became the Routh-Hurwitz stability criterion

Adolf Hurwitz

- 1859 – 1919
- German mathematician
- 1895 – Determined the Routh-Hurwitz stability criterion

Routh-Hurwitz stability criterion

- Stability information without the need to solve for the CL system poles
- How many CL system poles are in the LHP, RHP, and on the imaginary axis

2 steps
1. Generate Routh table
2. Interpret the Routh table

Figure: 1905 FIFA World Cup – Germany vs. England

Generating a basic Routh table, [1, p. 306]

Procedure

1. Label rows with powers of $s$ from the highest power of the denominator of the CLTF down to $s^0$
2. In the 1st row, horizontally list every other coefficient starting with the coefficient of the highest power of $s$
3. In the 2nd row, horizontally list every other coefficient starting with the coefficient of the next highest power of $s$

Table: Routh table

<table>
<thead>
<tr>
<th>$s^0$</th>
<th>$a_0$</th>
<th>$a_2$</th>
<th>$a_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^2$</td>
<td>$a_3$</td>
<td>$a_5$</td>
<td>$a_7$</td>
</tr>
</tbody>
</table>

Figure: Equivalent CL TF

4. Remaining row entries are filled with the negative determinant of entries in the previous 2 rows divided by entry in the 1st column directly above the calculated row. The left-hand column of the determinant is always the 1st column of the previous 2 rows, and the right-hand column is the elements of the column above and to the right.

Table: Routh table

<table>
<thead>
<tr>
<th>$s^0$</th>
<th>$a_0$</th>
<th>$a_2$</th>
<th>$a_4$</th>
<th>$a_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^2$</td>
<td>$a_3$</td>
<td>$a_5$</td>
<td>$a_7$</td>
<td>$a_9$</td>
</tr>
</tbody>
</table>

Figure: Equivalent CL TF
Interpreting a basic Routh table, [1, p. 307]

- The number of roots of the polynomial that are in the RHP is equal to the number of signs changes in the 1st column of a Routh table
- A system is stable if there are no sign changes in the first column of the Routh table

Special cases, [1, p. 308]

2 special cases
1. Zero only in the 1st column
   - If the 1st element of a row is a zero, division by zero would be required to form the next row
2. Entire row of zeros
   - Result of there being a purely even polynomial that is a factor of the original polynomial

Entire row of zeros, [1, p. 311]

- Purely even polynomials: Only have roots that are symmetrical and real
  - Root positions to generate even polynomials (symmetrical about the origin)
    1. Symmetrical and real
    2. Symmetrical and imaginary
    3. Quadrantal
  - Even polynomial appears in the row directly above the row of zeros

- Every entry in the table from the even polynomial’s row to the end of the chart applies only to the even polynomial
- Number of sign changes from the even polynomial to the end of the table equals the number of RHP roots of the even polynomial
- Even polynomial must have the same number of LHP roots as it does RHP roots
- Remaining poles must be on the imaginary axis
- The number of sign changes, from the beginning of the table down to the even polynomial, equals the number of RHP roots
- Remaining roots are LHP roots
- The other polynomial can contain no roots on the imaginary axis
6 Stability
- 6.1 Introduction
- 6.2 Routh-Hurwitz criterion
- 6.3 Routh-Hurwitz criterion: special cases
- 6.4 Routh-Hurwitz criterion: additional examples
- 6.5 Stability in state space

Some examples, [1, p. 314]

**Example (Standard)**
- **Problem:** Find the number of poles in the LHP, RHP, and on the imaginary axis
- **Solution:** On board
  - CLTF
  - Characteristic equation
  - Generate Routh table
  - Interpret Routh table

**Example (Zero in 1st column)**
- **Problem:** Find the number of poles in the LHP, RHP, and on the imaginary axis
- **Solution:** On board
  - CLTF
  - Characteristic equation
  - Generate Routh table
  - Epsilon method
  - Interpret Routh table

**Example (Zero in 1st column)**
- **Problem:** Find the number of poles in the LHP, RHP, and on the imaginary axis
- **Solution:** On board
  - CLTF
  - Characteristic equation
  - Generate Routh table
  - Reciprocal root method
  - Epsilon method
  - Interpret Routh table

Table: Routh table

<table>
<thead>
<tr>
<th>s^4</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^3</td>
<td>11</td>
</tr>
<tr>
<td>s^2</td>
<td>2</td>
</tr>
<tr>
<td>s</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ T(s) = \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200} \]

\[ CE(s) = s^4 + 6s^3 + 11s^2 + 6s + 200 \]
Some examples, [1, p. 314]

### Example (Zero in 1st column)

- **Problem:** Find the number of poles in the LHP, RHP, and on the imaginary axis
- **Solution:** On board
  - CLTF
  - Characteristic equation
  - Generate Routh table
  - Epsilon method
  - Interpret Routh table

**Table: Routh table**

<table>
<thead>
<tr>
<th>s^5</th>
<th>s^4</th>
<th>s^3</th>
<th>s^2</th>
<th>s^1</th>
<th>s^0</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td></td>
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<tr>
<td>2</td>
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</table>

\[ T(s) = \frac{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}{2s^4 + 3s^3 + 2s^2 + 3s + 2} \]

**RCCE(s) = s^5 + 2s^4 + 3s^3 + 2s^2 + 3s + 2**

---

### Example (Row of zeros)

- **Problem:** Find the number of poles in the LHP, RHP, and on the imaginary axis
- **Solution:** On board
  - CLTF
  - Characteristic equation
  - Generate Routh table
  - Interpret Routh table

**Table: Routh table**

<table>
<thead>
<tr>
<th>s^5</th>
<th>s^4</th>
<th>s^3</th>
<th>s^2</th>
<th>s^1</th>
<th>s^0</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
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</tr>
</tbody>
</table>

\[ T(s) = \frac{s^5 + 3s^4 + 10s^3 + 24s^2 + 48s + 96 + 128s^2 + 192s + 128}{s^4 + 3s^3 + 10s^2 + 24s^1 + 48s + 96 + 128s^2 + 192s + 128} \]

**CE(s) = s^5 + 3s^4 + 10s^3 + 24s^2 + 48s + 96 + 128s^2 + 192s + 128**

---

**Definition (eigenvalues, eigenvectors, & characteristic equation)**

- **Eigenvalues**, \( \lambda \), of system matrix, \( A \)
  - System poles
  - Values that permit a nontrivial solution (other than 0) for eigenvectors, \( x \), in the equation
    \[ Ax = \lambda x \]
    \[ x = (\lambda I - A)^{-1} 0 \]
    \[ = \frac{\text{adj}(\lambda I - A)}{\det(\lambda I - A)} 0 \]
  - All solutions will be the null vector except for the occurrence of zero in the denominator
  - This is the only condition where elements of \( x \) will be 0/0 or indeterminate, it is the only case where a nonzero solution is possible
  - Solutions of the characteristic equation \( \det(sI - A) = 0 \), a polynomial

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**Bibliography**