Chapter 12 – Design via State Space

12 Design via state space
- 12.1 Introduction
  - Can be applied to a wider class of systems than transform methods
  - Systems with nonlinearities
  - Multiple-input, multiple output (MIMO) systems
  - Systems of higher order than 2
- We will focus on the application to linear systems
  - Specify all CL poles
  - Parameters for each CL pole
  - Technique for finding these parameter values
- Cannot specify CL zero locations
- Sensitive to parameter changes
- Wide range of computational support
  - Loss of graphic insight into a design problem
12 Design via state space
12.2 Controller design

State-variable FB control, [1, p. 665]

Concept
- $n^{th}$-order CL characteristic equation (CE)
  \[
  \det(sI - A) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 = 0
  \]
- There are $n$ coefficients whose values determine the $n$ CL poles
- Introduce $n$ adjustable parameters into the system and relate them to the $n$ coefficients, so that we can place the $n$ CL poles

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State-variable FB control, [1, p. 666]

Concept
- Before, output-variable FB, now, state-variable FB
  - Each state variable is fed back to the control, $u$, through a gain, $k_i$
  - State-variable FB gain: $-K$
- CL system is plant with state-variable FB

\[
\dot{x} = Ax + Bu \\
= Ax + B(-Kx + r) \\
= (A - BK)x + Br \\
y = Cx
\]

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State-variable FB control in PV form, [1, p. 668]

Concept
- Pole placement for plants in phase-variable (PV) form
  1. Represent the plant in PV form
  2. FB each PV to the input of the plant through a gain, $k_i$
  3. Find the CE for the CL system
  4. Decide upon all CL pole locations and determine equivalent CE
  5. Equate like coefficients of the CE and solve for $k_i$

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State-variable FB control in PV form, [1, p. 668]

**Concept**
- State-variable FB
  \[ u = -Kx; \quad K = [k_1 \quad k_2 \ldots \quad k_n] \]
- CL system
  \[
  A - BK = \begin{bmatrix}
  0 & 1 & 0 & \ldots & 0 \\
  0 & 0 & 1 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  -(a_0 + k_1) & -(a_1 + k_2) & -(a_2 + k_3) & \ldots & -(a_{n-1} + K_n)
  \end{bmatrix}
  \]
- CL system CE
  \[
  \det(sI - (A - BK)) = s^n + (a_{n-1} + k_n)s^{n-1} + \ldots + (a_1 + k_2)s + (a_0 + k_1) = 0
  \]

Example, [1, p. 669]

**Example (Controller design for PV form)**
- **Problem:** Design the PV FB gains to yield
  - %OS = 9.5%
  - \( T_s = 0.74 \) seconds
- **Solution:** On the board

**Definitions, [1, p. 672]**

- **Controllability:** If an input to a system can be found that takes every state variable from a desired initial state to a desired final state, the system is said to be controllable; otherwise the system is uncontrollable.
  - Control variable, \( u \), can be used to control the behavior of each state variable in \( x \)
  - Poles of the control system can be placed where we desire
  - Determine, a priori, whether pole placement is a viable design technique for a controller

**Concept**
- Desired CL system CE
  \[
  \det(sI - (A - BK)) = s^n + d_{n-1}s^{n-1} + \ldots + d_1s + d_0 = 0
  \]
  \[ d_i = a_i + k_{i+1}; \quad i = 0, 1, 2, \ldots, n - 1 \]
- CL system TF
  - Denominator polynomial: the CE
  - Numerator polynomial: formed from the coefficients of the output coupling matrix, \( C \), for systems written in PV form
  - Same for plant and CL system

**Controllability by inspection, [1, p. 673]**

- **Concept**
  - When the system matrix, \( A \), is in diagonal or parallel form, it is apparent whether or not the system is controllable
    - A system with distinct (no repeat) eigenvalues and a diagonal system matrix, \( A \), is controllable if the input coupling matrix, \( B \), does not have any rows that are zero
The controllability matrix, [1, p. 674]

- In other forms, the existence of paths from the input to the state variables is not a criterion for controllability since the equations are not decoupled
  - \( n \)-th order plant
    \[ \dot{x} = Ax + Bu \]
  - is completely controllable if the matrix
    \[ CM = [B \ AB \ \ldots \ \ A^{n-1}B] \]
  - is of rank \( n \), where \( CM \) is called the controllability matrix

Example, [1, p. 675]

Example (Controllability via the controllability matrix)

- Problem: Determine if the system is controllable
  \[ A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \]
- Solution: On the board

Alternative approaches to controller design, [1, p. 676]

- For systems not represented in PV form
- 2 approaches
  - Matching coefficients: Matching the coefficients of
    \[ \det(sI - (A - BK)) \]
    with coefficients of the desired CE
  - Same method used for systems in PV representation
  - Transformation: Transforming the system to PV form, designing the control FB gain, & transforming the designed system back to its original state-variable representation

Approach – matching coefficients, [1, p. 677]

Concept
- Matching the coefficients of
  \[ \det(sI - (A - BK)) \]
  with coefficients of the desired CE
- Leads to difficult calculations of the control gains, especially for higher-order systems not represented with PVs
- Problem: Design state-variable control FB gain for the plant to yield
  - \( %OS = 15\% \)
  - \( T_s = 0.5 \) second
  \[ G(s) = \frac{10}{(s + 1)(s + 2)} \]
- Solution: On the board
**Approach – transformation, [1, p. 678]**

**Procedure**

1. Transform the system to PV representation
2. Design the state-variable control FB gain
3. Transform the system in PV representation back to the original representation

**Example (Controller design by transformation)**

**Problem:** Design state-variable control FB gain for the plant to yield

- \( \% \text{OS} = 20.8\% \)
- \( T_s = 4 \text{ seconds} \)

\[
G(s) = \frac{s + 4}{(s + 1)(s + 2)(s + 5)}
\]

**Solution:** On the board
Observer motivation, [1, p. 683]

Concept

- Controller design relies upon access to the state variables for FB through adjustable gains
- Estimate states can be fed to the controller
- Observer: Estimator used to calculate state variables that are not accessible from the plant
- Observer error:
  \[ e_x = x - \hat{x} \]
  \[ y - \hat{y} = C(x - \hat{x}) = Ce_x \]

- Plant:
  \[ \dot{x} = Ax + Bu \]
  \[ y = Cx \]

Observer canonical form yields an easy solution for the observer FB gain
- Observer FB gain, \( L \): Ensures the TR of the observer is faster than the response of the controlled loop in order to yield a rapidly updated estimate of the state vector
12 Design via state space
12.5 Observer design

Procedure
1. Find error system, i.e., state equations for error between actual state vector & estimated state vector,
\[ \dot{x} = A\hat{x} + Bu + L(y - \hat{y}) \]
\[ \hat{y} = C\hat{x} \]
2. Find CE for error system
3. Evaluate required observer FB gain, \( L \), to meet rapid TR for observer
4. Select eigenvalues of observer to yield stability & desired TR that is faster than controlled CL response

Definitions, [1, p. 690]

- **Observability**: If the initial-state vector, \( x(t_0) \), can be found from inputs, \( u(t) \), and measurements, \( y(t) \), over a finite interval of time from \( t_0 \), the system is said to be observable; otherwise the system is said to be unobservable.
  - Knowledge of measured output variables, \( y \), and control inputs, \( u(t) \), can be used to observe the behavior of each state variable in \( x \)
  - Poles of the observer system can be placed where we desire
  - Determine, a priori, whether pole placement is a viable design technique for an observer

The observability matrix, [1, p. 691]

- In other forms, the existence of paths from the output to the state variables is not a criterion for observability since the equations are not decoupled
  - \( n^{th} \)-order plant
  \[ \dot{x} = Ax + Bu \]
  \[ y = Cx \]
  is completely observable if the matrix
  \[ O_M = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \]
  is of rank \( n \), where \( O_M \) is called the observability matrix

Example (Observability via the observability matrix)

- **Problem**: Determine if the system is observable
  \[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad C = [0 & 5 & 1] \]
  
- **Solution**: On the board
Example (Unobservability via the observability matrix)

- **Problem:** Determine if the system is observable

\[
A = \begin{bmatrix}
0 & 1 \\
-5 & -41
\end{bmatrix}; \quad B = \begin{bmatrix}
0 \\
1
\end{bmatrix}; \quad C = \begin{bmatrix}
5 & 4
\end{bmatrix}
\]

- **Solution:** On the board

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Example (Observer design by matching coefficients)

- **Problem:** Design an observer FB gain for the system in PV representation with a TR described by

\[
\zeta = 0.7 \quad \omega_n = 100
\]

\[
G(s) = \frac{407(s + 0.916)}{(s + 1.27)(s + 2.69)}
\]

- **Solution:** On the board

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Alternative approaches to controller design, [1, p. 676]

**Concept**

- For systems not represented in observer canonical form
- 2 approaches
  - **Matching coefficients:** Matching the coefficients of
    \[
    \det(sI - (A - LC))
    \]
    with coefficients of the desired CE
  - **Transformation:** Transforming the system to observer canonical form, designing the observer FB gain, & transforming the designed system back to its original state-variable representation

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Approach - matching coefficients, [1, p. 693]

**Concept**

- Matching the coefficients of
  \[
  \det(sI - (A - LC))
  \]
  with coefficients of the desired CE
- Leads to difficult calculations of the observer FB gain, especially for higher-order systems not in PV representation

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Approach – transformation, [1, p. 695]

**Procedure**

1. Transform the system to PV representation
2. Design the observer FB gain
3. Transform the system in PV representation back to the original representation
Approach – transformation, [1, p. 695]

Procedure

1. Transform the system to PV representation
   - Plant not in PV representation
     \[
     \dot{z} = Az + Bu \\
     y = Cz
     \]
     with observability matrix
     \[
     OM_z = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}
     \]

2. Design the observer FB
   - Transformed plant with FB gains
     \[
     \dot{e}_x = (P^{-1}AP - L_xCP)e_x \\
     y - \hat{y} = CPe_x
     \]

Example, [1, p. 695]

Example (Observer design by transformation)

- **Problem:** Design an observer in cascade form

  \[
  G(s) = \frac{1}{(s + 1)(s + 2)(s + 5)}
  \]

- **Solution:** On the board
Concepts, [1, p. 700]

**Concept**
- Design systems in state-space representation for steady-state error
- Error is fed forward to the controlled plant via an integrator
  - Additional state variable
    \[ \dot{x}_N = r -Cx \]
- Plant
  \[ \dot{x} = Ax + Bu \]
  \[ \dot{x}_N = -Cx + r \]
  \[ y = Cx \]
- Control FB
  \[ u = -Kx + Ke x_N \]
  \[ = -[K -Ke] \begin{bmatrix} x \\ x_N \end{bmatrix} \]

Example, [1, p. 701]

**Example (Design of integral control)**
- **Problem:**
  1. Design a controller without integral control to yield
     - \( \%OS = 10\% \)
     - \( Ts = 0.5 \) second
     Evaluate the steady-state error for a unit step
  2. Repeat the design using integral control. Evaluate the steady-state error for a unit step input.
- **Solution:** On the board

Bibliography