Note Title 10/29/2007

Fall 2007
Midterm #2 Solution

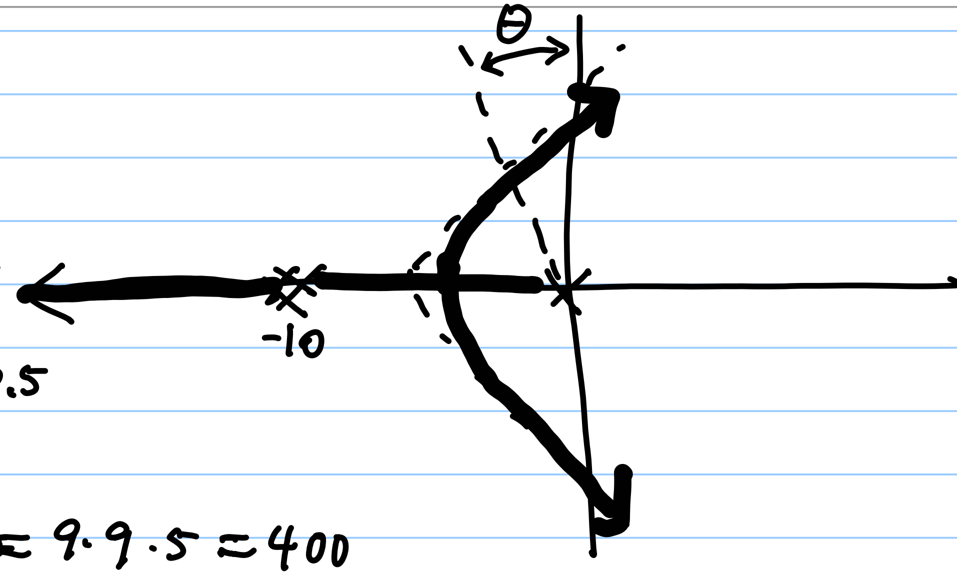
(1.a) $G(s) = \dfrac{1}{s(s+10)^2}$

$\sigma = \dfrac{-10-10}{3} = -6.67$

$\sin(\theta) = \zeta = 0.5$

$\theta = 30^\circ$

$K = l_1 \cdot l_2 \cdot l_3 = 9 \cdot 9 \cdot 5 = 400$



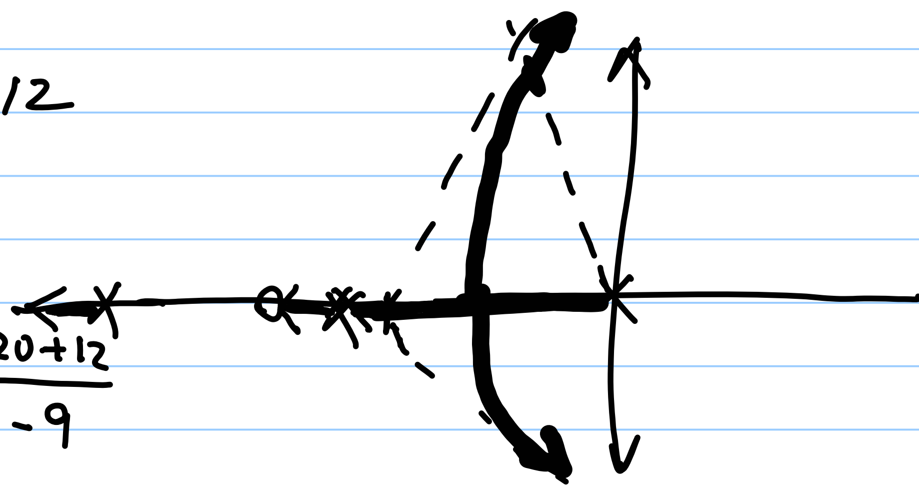
(1.b)

let zero $= -12$

pole $= -20$

$\Rightarrow$ lead comp.

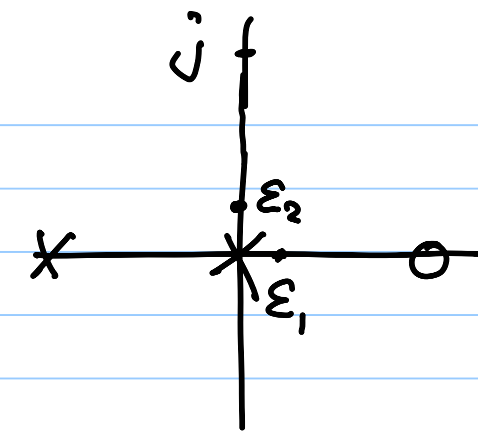
$\sigma = \dfrac{-10-10-20+12}{3} = \dfrac{-28}{3} = -9$

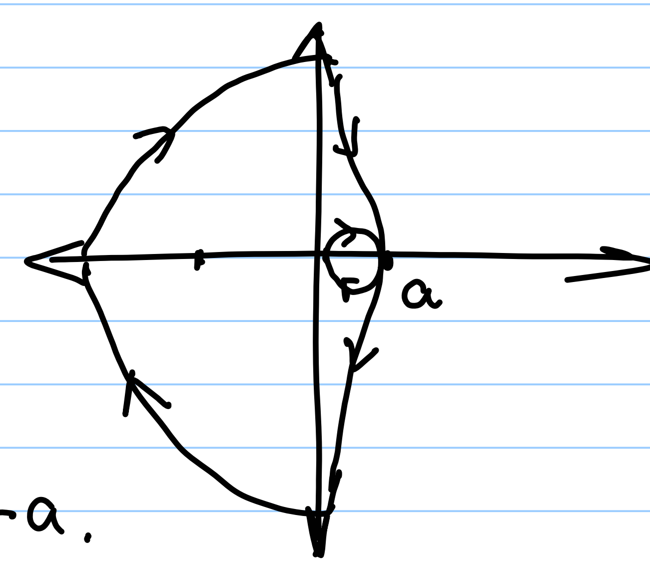


(2)

$$G(s) = \frac{s-1}{s(s+1)}$$

freq	mag	phase
ε_1	∞	$180°$
ε_2	∞	$90°$
j	1	$0°$
$2j$	$1/2$	
∞j	0	$-90°$





(2b) The closed loop system is unstable for $K > 0$ (1 RHP pole) or $K < -a$ (2 RHP poles).

The system is stable for $0 > K > -a$.

(3) $(\lambda+2)(\lambda+3)=\lambda^2+5\lambda+6$

$A=\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}$, $A^2=\begin{bmatrix}5 & -4\\ -4 & 5\end{bmatrix}$, $Ab=\begin{pmatrix}1\\ -2\end{pmatrix}$

$$K=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & -2\end{bmatrix}^{-1}\left(\begin{bmatrix}5 & -4\\ -4 & 5\end{bmatrix}+5\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}+\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}\right)$$

$$=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1 & 0.5\\ 0 & -0.5\end{bmatrix}\begin{bmatrix}16 & -14\\ -14 & 16\end{bmatrix}=\begin{bmatrix}0 & -0.5\end{bmatrix}\begin{bmatrix}16 & -14\\ -14 & 16\end{bmatrix}$$

$$=\begin{bmatrix}7 & -8\end{bmatrix}$$

(4) $$(sI-A)^{-1} = \begin{pmatrix} s-1 & 2 \\ 2 & s-1 \end{pmatrix}^{-1} = \frac{1}{s^2-2s-3}\begin{bmatrix} s-1 & -2 \\ -2 & s-1 \end{bmatrix}$$

$$c(sI-A)^{-1}b = \frac{1}{s^2-2s-3}\begin{bmatrix} 0 & 4 \end{bmatrix}\begin{bmatrix} s-1 & -2 \\ -2 & s-1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s-3)} 4(s-3) = \frac{4}{s+1}$$

The mode associated with the eigenvalue -1 is the mode that is both controllable and observable.
The mode associated with eigenvalue 3 is either uncontrollable or unobservable.

(4) (alternative solution)

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-1 & 2 \\ 2 & \lambda-1 \end{bmatrix} = (\lambda-1)^2 - 4 = \lambda^2 - 2\lambda + 1 - 4$$

$$= \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$$

$$\lambda_1 = -1, \quad \lambda_2 = 3$$

$$(A - \lambda_1 I)V_1 = 0 \qquad \begin{pmatrix} 1-\lambda_1 & -2 \\ -2 & 1-\lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$V_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(A - \lambda_2 I)V_2 = 0 \qquad \begin{pmatrix} 1-3 & -2 \\ -2 & 1-3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

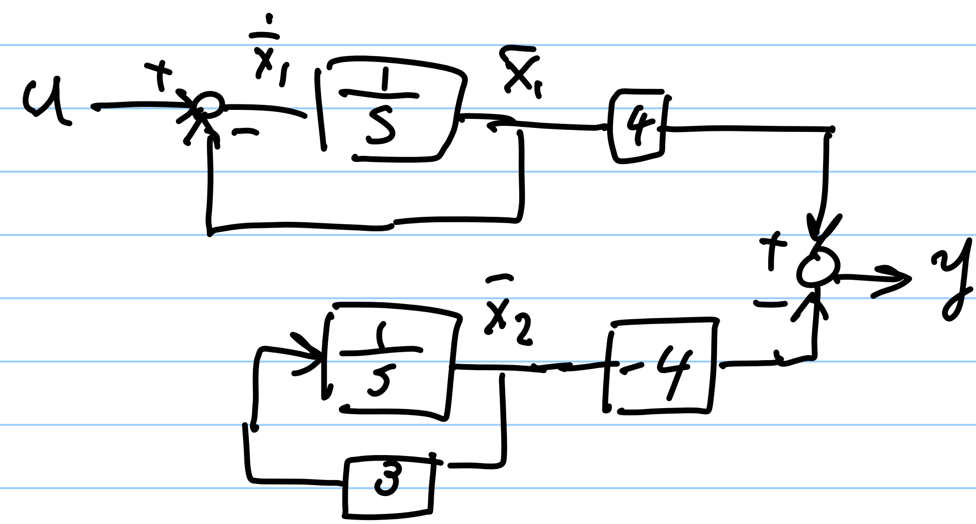
$$V_2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Let $T = [V_1 \vdots V_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $T^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\tilde{b} = T^{-1}b = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\tilde{c} = cT = \begin{bmatrix} 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \end{bmatrix}$$



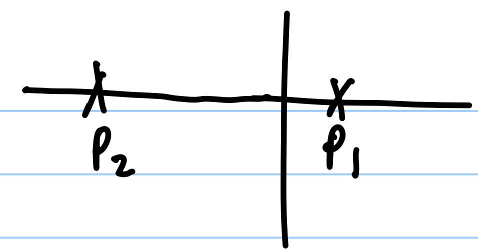
The mode associated with $\lambda = -1$ is controllable and observable.
The mode associated with $\lambda = 3$ is not controllable but observable.

(5) If $|G(j\omega)| < 1$, for all ω, its Nyquist plot will not encircle the -1 point. Since it is also open loop stable, # of RHP pole $= 0$.

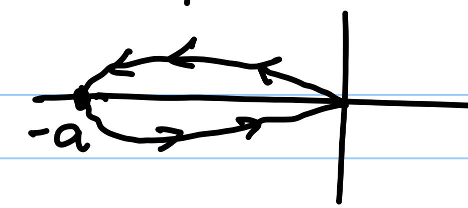
$$Z = N + P = 0 + 0 = 0$$

The closed loop system is stable!

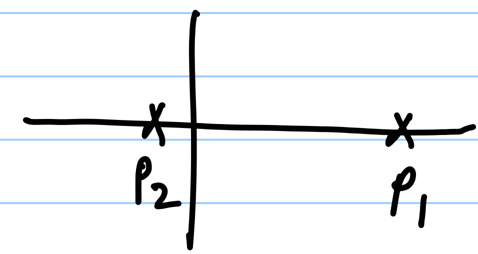
(6) If $|p_2| > |p_1|$

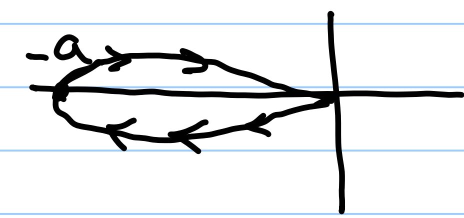


$a = \frac{1}{|p_1 \cdot p_2|}$



If $|p_2| < |p_1|$





(6.a) If $|p_2| > |p_1|$, only K is enough. CCW encirclement of $-1/K$ can be made without using $C(s)$.

(6.b) If $\frac{1}{|p_1 p_2|} > 1$, the system can be stabilized by only $C(s)$.

(6.c) no restriction on p_1 or p_2.