EE128 - FALL 2005
MIDTERM REVIEW NOTES

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1. Introduction

These notes cover some important topics for the midterm. It is not meant to be a substitute for doing the readings from the textbook, nor is it meant to be an exhaustive list of all the material that will appear on the midterm. The goal is to present some of the material that you have already seen in a different manner.

2. State-Space System Representation

A very powerful and very general, mathematical model of a system is the state-space representation. Intuitively speaking, the state of a system is a collection of variables that tell us how much "energy" is currently in the system. Taking this intuition further, this means that the outputs of the system are completely characterized by both its states and inputs; the same input, but with different initial states, will lead to different outputs. In this class, we will only be concerned with a less general, state-space representation of the form:

$$\dot{x} = f(x, u)$$

Four important characteristics about the state-space representation above are:

(1) In general, $\dot{x}$, x, and u are vectors.
(2) The system is time-invariant; specifically, the vector field pushing x is not a function of time.
(3) There are no derivatives higher than a first derivative.
(4) The system is not necessarily linear.

The last bullet is an important point, and needs further discussion. If the system is linear, it admits a simpler set of equations. More importantly, it is much easier to analyze a linear, time-invariant (LTI) system than it is to analyze a non-linear system. If the system is linear (and time-invariant, by the assumptions made in class), then our state-space representation is:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where A, B, C, and D are matrices of the appropriate dimensions. Suppose that x is a column-vector of dimension $n_\Sigma \times 1$, u is a vector of dimension $n_i \times 1$, and y is a vector of dimension $n_o \times 1$. Then, the matrices A, B, C, and D will respectively be of dimension $n_\Sigma \times n_\Sigma$, $n_\Sigma \times n_i$, $n_o \times n_\Sigma$, and $n_o \times n_i$.

2.1. **Example: Non-Linear System.**

The simplified equations of motion for a ball that is magnetically levitated by an electro-magnet is:

$$m\ddot{r} = mg - \frac{ki}{r^2}$$

where m is the mass of the ball, r is the distance of the ball from the electro-magnet, i is the current provided to the electro-magnet, g is the gravitational constant, and k is a constant which incorporates the physical constants of the electro-magnet. This equation is simply a statment that the force on the ball is the sum of a downward force due to gravity and an upward force due to the magnetic field. Note that we must take the negative of this overall force because of the way that we have defined r. To put this into our state-space form, we define $x_1 \equiv r$, $x_2 \equiv \dot{r}$, and $u_1 \equiv i$. Thus, our system can be rewritten as:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ g - \frac{ku_1}{mx_1^2} \end{pmatrix}$$

3. Linearization

It is often the case that a system that we wish to analyze or control is non-linear. One technique for handling such systems is to consider perturbations (small-changes) about a nominal trajectory. Such perturbations can be reasonably approximated by considering the linear terms of the Taylor expansion of the system representation. In this class, we will only be concerned with perturbations about equilibrium points.

More succintly, suppose that our system representation is:

$$\dot{x}(t) = f(x(t), u(t))$$

The equilibrium points occur when $\dot{x}(t) = 0$, so denote these points as $(x_{0_i}, u_{0_i}) \in \{(x,u)|f(x,u) = 0\}$. Note that in general, we will have multiple equillibrium points, and so we can individually linearize the system about each of these points. The linearization about the equillibrium point (x_{0_i}, u_{0_i}) is:

$$\dot{x} = \left. J_x \right|_{(x_{0_i}, u_{0_i})} x + \left. J_u \right|_{(x_{0_i}, u_{0_i})} u$$

where J_x and J_u are defined as follows:

$$J_x = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_{n_\Sigma}} \\ \frac{\partial f_2}{\partial x_1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_{n_\Sigma}}{\partial x_1} & \cdots & & \frac{\partial f_{n_\Sigma}}{\partial x_{n_\Sigma}} \end{pmatrix}$$

$$J_u = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_{n_i}} \\ \frac{\partial f_2}{\partial u_1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_{n_\Sigma}}{\partial u_1} & \cdots & & \frac{\partial f_{n_\Sigma}}{\partial u_{n_i}} \end{pmatrix}$$

3.1. **Example: Non-Linear System (cont.)** We will linearize the magnetically levitating ball system described in the previous example. First, we must determine the equilibrium points of the system, that is the $(x_{0_i}, u_{0_i}) \in \{(x,u)|f(x,u) = 0\}$. Here, we will consider a special case in which we wish to set $x_1 = C$. This condition means that we would like the distance between the electro-magnet and the ball to be C. Thus, we have:

$$f(x,u) = 0 \Rightarrow \begin{pmatrix} x_2 \\ g - \frac{ku_1}{mx_1^2} \end{pmatrix} = \begin{pmatrix} x_2 \\ g - \frac{ku_1}{mC^2} \end{pmatrix} = 0$$

This occurs when $x_{0_1} = (x_1, x_2)^T = (C, 0)^T$ and $u_{0_1} = u_1 = \frac{mgC^2}{k}$. Next, we calculate the Jacobian matrices associated with this system and then substitute in our equilibrium points:

$$J_x|_{(x_{0_1}, u_{0_1})} = \begin{pmatrix} 0 & 1 \\ \frac{2ku_1}{mx_1^3} & 0 \end{pmatrix}\Bigg|_{(x_{0_1}, u_{0_1})} = \begin{pmatrix} 0 & 1 \\ \frac{2g}{C} & 0 \end{pmatrix}$$

$$J_u|_{(x_{0_1}, u_{0_1})} = \begin{pmatrix} 0 \\ \frac{-k}{mx_1^2} \end{pmatrix}\Bigg|_{(x_{0_1}, u_{0_1})} = \begin{pmatrix} 0 \\ \frac{-k}{mC^2} \end{pmatrix}$$

Therefore, the system linearized about the equilibrium point for $x_1 \equiv r = C$ is:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ \frac{2g}{C} & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{-k}{mC^2} \end{pmatrix} u$$

For this particular equation, we can easily convert it into a more familiar form. The above state-space representation for the linearized system can be written as $\ddot{r} - \frac{2g}{C} r = \frac{-k}{mC^2} i$. This is a system representation that we know how to analyze and control, unlike the non-linear system that we originally started with.

4. Stability

The concept of stability is crucial to the control engineer. Intuitively, the idea of system stability is simple. Unfortunately, at a practical and technical level the meaning of stability becomes unclear.

There are three major types of stability that we will be concerned with in this class: Bounded-Input Bounded-Output (BIBO) stability, marginal stability, and asymptotic stability. A system is said to be BIBO stable if an input that is bounded yields an output that is also bounded. More formally, suppose that we have a linear, time-invariant system $y(t) = \mathcal{L}(u)$, with input u and output y. This system is BIBO stable if and only if,

$$\text{given } u(t) \text{ and } M_u \ni |u(t)| < M_u \ \forall\ t,\ \exists\ M_y \ni |\mathcal{L}(u)| = |y(t)| < \ M_y \ \forall\ t$$

This formal definition can be used to derive several important criterion for BIBO stability of LTI systems. An LTI system is BIBO stable if and only if the impulse response of the LTI system is absolutely integrable. Also, an LTI system is BIBO stable if and only if all of its poles are in the Open Left-Half Plane (OLHP). It is important to note that poles on the $j\omega$-axis mean that a system is *not* BIBO stable.

An interesting point to note is that the existence of the Fourier transform of an impulse response is *not* enough to guarantee that a system is BIBO stable. An important and surprising example of this is the ubiqutous sinc function ($h(t) = \frac{\sin \pi t}{\pi t}$) which has a Fourier transform but is *not* BIBO stable.

Marginal stability is not the same as BIBO stability. A system is marginally stable if it has a bounded output for some bounded inputs, but unbounded outputs for certain bounded inputs. In terms of an LTI system, a system with poles on the $j\omega$-axis is marginally stable. This is because the system output is unbounded if we put in a sinusoid whose frequency matches those of the poles on the imaginary axis. Intuitively, we can think of this in terms of the mechanical concept of resonance. If we vibrate an object at its resonance frequency, the object will begin to oscillate violently. There are many examples of this, including examples of bridges that have collapsed because the wind caused the bridge to vibrate at its resonant frequency.

The last type of stability that we will deal with is asymptotic stability. A system is asymptotically stable if its impulse response goes to zero as time goes to infinity. For an LTI system, a system is asymptotically stable if and only if it is BIBO stable. Thus, asymptotic stability and BIBO stability are equivalent for LTI systems.

5. Nyquist Stability Criterion

The Nyquist stability criterion is surprisingly one of the easier stability criterions to prove, despite the fact that it requires complex analysis techniques to prove. Fortunately, it does have a fairly intuitive explanation of why it works and what it does. It is important to keep in mind that the domain and range are seperate entities, and to keep these ideas mentally seperate. We will refer to the contour in the domain of $G(s)$ as $d(s)$. And, we will refer the resulting contour in the range of $G(s)$ as $r(s)$.

Suppose that we evaluate $G(s)$ about a contour $r(s)$. The number of encirclements of $r(s)$ about the origin depends on the number of poles and zeros contained within $d(s)$. Each additional pole inside of $d(s)$ makes $r(s)$ encircle the origin -1 times around the origin, and each additional zero inside of $d(s)$ makes $r(s)$ encircle the origin 1 time around the origin. A Nyquist plot is simply the evaluation of $G(s)$ along a contour $d(s)$ that completely contains the RHP. Thus, by counting the number of encirclements, we can determine the number of zeros minus the number of poles that are contained in the RHP.

Now, the transfer function of the canonical feedback gain system is $\frac{Y(s)}{U(s)} = \frac{G(s)}{1+KG(s)}$. Pretend that we fix K to a certain value; we can make a Nyquist plot of this function. Call the image of the Nyquist plot $r(s)$. The number of encirclements of $r(s)$ about the origin will be the number of zeros of $1 + KG(s)$ in the RHP plan minus the number of poles of $1 + KG(s)$ in the RHP. However, the poles of $1 + KG(s)$ are simply the poles of the $G(s)$. Thus, by counting the number of encirclements of $r(s)$ about the origin and then adding the number of poles of $G(s)$ in the RHP, we can calculate the number of zeros of $1 + KG(s)$ in the RHP.

This is important because the zeros of $1+KG(s)$ are the poles of the transfer function $\frac{Y(s)}{U(s)} = \frac{G(s)}{1+KG(s)}$. For stability, we require no poles in the RHP. This is equivalent to saying that we require no zeros of $1 + KG(s)$ in the RHP. We can get the information about the zeros in the RHP from the Nyquist plot of $1 + KG(s)$. However, there is a simplifcation that we can make to the procedure. The Nyquist plot of $1 + KG(s)$ is mereley the Nyquist plot of $G(s)$ that has been translated to the left by $-\frac{1}{K}$. Therefore, we can just do a Nyquist plot of $G(s)$ and count the encirclements about $-\frac{1}{K}$. Counting the number of encirclements of $r(s)$ about $s = -\frac{1}{K}$ and then adding the number of poles of $G(s)$ in the RHP, we can calculate the number of zeros of $1 + KG(s)$ in the RHP.

To make a Nyquist plot, we evaluate $G(s)$ along a clockwise countour through the RHP. We must make sure to have the contour avoid any poles lying on the $j\omega$-axis, and the portion of the contour not on the axis must be at infinity. To do this, we seperate the contour into simple curves. These curves take the form of either straight lines ($c_1 = j\omega$, for some range of ω) or arcs ($c_2 = re^{j\theta}$, for some value of r and some range of θ). The arcs are either of "large" radius or "small" radius. This means, that when you do the plot you can arbitrarily choose some large and some small radius, depending on the need. If the arc spans the RHP, then the radius should be large. If the arc is about the $j\omega$-axis around a pole, then the radius should be small. The Nyquist plot is then the combination of the image of $G(s)$ evaluated alone each of these simple curves.

For the most part, the transfer functions that we are dealing with are well-behaved. More rigorously, the transfer functions are proper or, more commonly, strictly proper. This means that as the frequency goes to infinity, the transfer function converges to either a constant value or to zero, respectively. Also, we usually do not have poles along the $j\omega$-axis. If this is the case, then we can actually draw a Nyquist plot from a Bode plot. A Bode plot is mereley the transfer function evaluated over the positive $j\omega$-axis. This is essentially one-half of the Nyquist plot. To get the Nyquist plot from a Bode plot, we take successive points on the Bode plot and plot the magnitude and phase as a complex number on the complex-plane. We let the points range from frequency 0 to infinity. However, this is only one half of the Nyquist plot. We must mirror the plot about the real-axis to generate the full Nyquist plot. An example will be instructive.

5.1. **Nyquist/Bode Plot Example.** Consider the low-pass filter transfer function:

$$\frac{Y(s)}{U(s)} = \frac{1}{1+s}$$

The Bode plot is easy to draw in this case, and is shown in Figure 1.

To calculate the Nyquist plot, we split our contour into three portions:

(1) $c_1 = j\omega$, for ω ranges from 0 to infinty
(2) $c_2 = re^{j\theta}$ for r large and θ ranges from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$
(3) $c_3 = j\omega$, for ω ranges from negative infinity to 0

Next, we evaluate $G(s)$ alone each of these three curves individually. We start with c_1:

$$G(s)\big|_{c_1} = \frac{1}{1+j\omega} \Rightarrow |G(s)|_{c_1} = \frac{1}{\sqrt{1+\omega^2}}$$
$$\angle\, G(s)\big|_{c_1} = -tan^{-1}\omega$$

Along c_2, we have r large, and we get:

$$G(s)\big|_{c_2} = \frac{1}{1+re^{j\theta}} \Rightarrow |G(s)|_{c_2} \approx \frac{1}{r}$$
$$\angle\, G(s)\big|_{c_2} \approx -\theta$$

Along c_3, we have:

$$G(s)\big|_{c_3} = \frac{1}{1-j\omega} \Rightarrow |G(s)|_{c_3} = \frac{1}{\sqrt{1+\omega^2}}$$
$$\angle\, G(s)\big|_{c_3} = tan^{-1}\omega$$

Plotting each of these segments over the corresponding ranges of ω and θ yields the Nyquist plot shown in Figure 2.

However, we can determine the Nyquist plot from the Bode plot. Observe how the Nyquist plot along c_2 is essentially of magnitude zero, since we have r large. Thus, the Nyquist plot is completeley characterized by the Nyquist plot along c_1 and c_3. But, there is something interesting with this fact. The Nyquist plot along c_1 is simply the Bode plot, and the Nyquist plot along c_3 is the mirror image of the Bode plot.

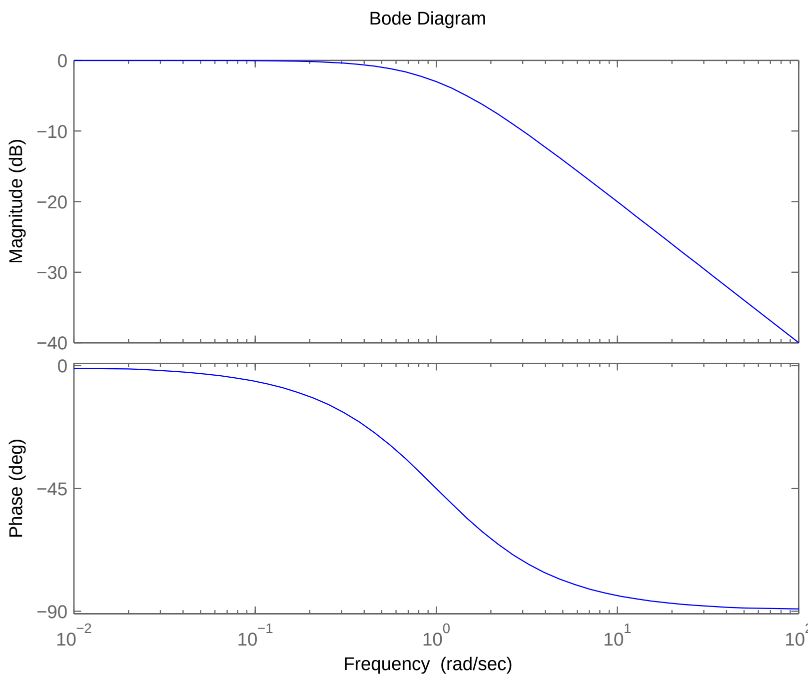

FIGURE 1. Bode Plot of $\frac{Y(s)}{U(s)}$

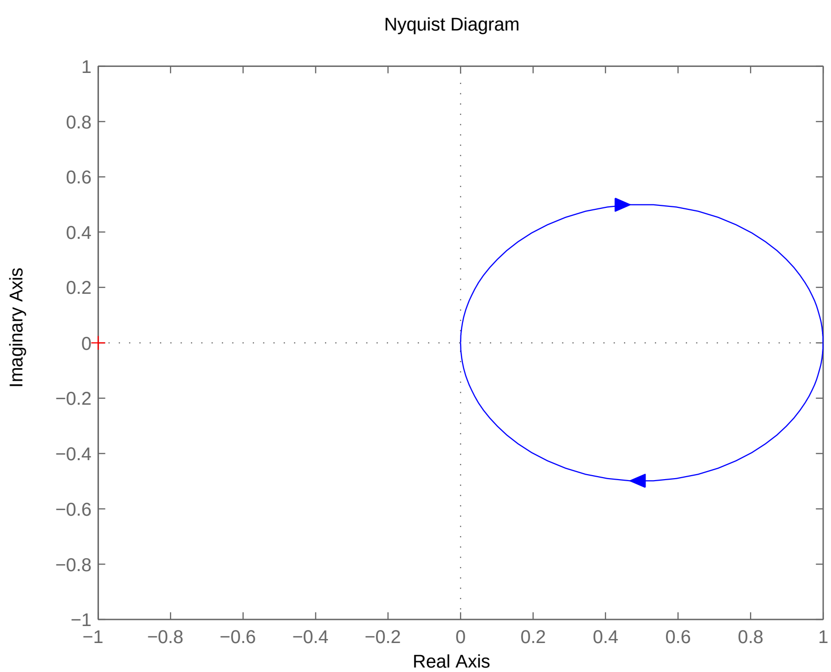

FIGURE 2. Nyquist Plot of $\frac{Y(s)}{U(s)}$