1. Magic with constraints
In this question, we will represent a problem in two different ways and show that strong duality holds in one case but doesn’t hold in the other.

Let

\[ f_0(x) = \begin{cases} 
  x^3 - 3x^2 + 4, & x \geq 0 \\
  -x^3 - 3x^2 + 4, & x < 0 
\end{cases} \]

(a) Consider the minimization problem

\[ p^* = \inf_{x \in \mathbb{R}} f_0(x) \quad \text{s.t.} \quad -1 \leq x, \ x \leq 1. \]

i. Show that \( p^* = 2 \) and the set of optimizers \( x \in \lambda^* \) is \( \lambda^* = \{-1, 1\} \) by examining the “critical” points, i.e., points where the gradient is zero, points on the boundaries, and \( \pm \infty \).

\[ x = \pm \infty, \pm 2 \]

\[ p^* = 2 \quad \text{attained at} \quad x \in \{-1, 1\}. \]

ii. Show that the dual problem can be represented as

\[ d^* = \sup_{\lambda_1, \lambda_2 \geq 0} g(\bar{\lambda}), \]

where

\[ g(\bar{\lambda}) = \min \left\{ g_1(\bar{\lambda}), g_2(\bar{\lambda}) \right\}, \]

with

\[ g_1(\bar{\lambda}) = \inf_{x \geq 0} x^3 - 3x^2 + 4 - \lambda_1(x + 1) + \lambda_2(x - 1), \]

\[ g_2(\bar{\lambda}) = \inf_{x < 0} -x^3 - 3x^2 + 4 - \lambda_1(x + 1) + \lambda_2(x - 1). \]

\[ g(\bar{\lambda}) = \inf_{x \geq 0} \bar{f}(x, \bar{\lambda}) \quad \text{and} \quad \inf_{x < 0} \bar{f}(x, \bar{\lambda}) \]

\[ = \inf \left( \min_{x \geq 0} \bar{f}(x, \bar{\lambda}), \quad \inf_{x < 0} \bar{f}(x, \bar{\lambda}) \right) \]
iii. Next, show that
\[ \begin{align*}
g_1(\lambda) & \leq -3\lambda_1 + \lambda_2 \\
g_2(\lambda) & \leq \lambda_1 - 3\lambda_2.
\end{align*} \]
Use this to show that \( g(\lambda) \leq 0 \) for all \( \lambda_1, \lambda_2 \geq 0 \).
\[ \begin{align*}
x = 2 & \Rightarrow g_1(\lambda) \leq -3\lambda_1 + \lambda_2 \\
x = -2 & \Rightarrow g_2(\lambda) \leq \lambda_1 - 3\lambda_2
\end{align*} \]
iv. Show that \( g(\bar{0}) = 0 \) and conclude that \( d^* = 0 \).
\[ \begin{align*}
g(\bar{0}) = \max_{\lambda} \min \{ g_1(\bar{0}), g_2(\bar{0}) \} = \min \{ \inf_{x \in \mathbb{R}} x^3 - 3x^2 + 4, \inf_{x \in \mathbb{R}} -x^3 - 3x^2 + 4 \} \\
= \min \{ 0, 0 \} = 0 \Rightarrow d^* = \frac{2}{\lambda}
\]
v. Does strong duality hold?
\[ \begin{align*}
\rho^* = 2 & \Rightarrow \text{NO!} \\
d^* = 0
\end{align*} \]

(b) Now, consider a problem equivalent to the minimization in (1):
\[ p^* = \inf_{x \in \mathbb{R}} f_0(x) \quad \text{s.t. } x^2 \leq 1 \]  
\[ (2) \]
Observe that \( p^* = 2 \) and the set of optimizers \( x \in \mathcal{X}^* \) is \( \mathcal{X}^* = \{-1, 1\} \), since this problem is equivalent to the one in part (a).

i. Show that the dual problem can be represented as
\[ d^* = \sup_{\lambda \geq 0} g(\lambda), \]
where
\[ g(\lambda) = \min(g_1(\lambda), g_2(\lambda)), \]
with
\[
\begin{align*}
g_1(\lambda) &= \inf_{x \geq 0} x^3 - 3x^2 + 4 + \lambda(x^2 - 1) \quad h(x, \lambda), \\
g_2(\lambda) &= \inf_{x < 0} -x^3 - 3x^2 + 4 + \lambda(x^2 - 1).
\end{align*}
\]

ii. Show that \( g_1(\lambda) = g_2(\lambda) = \begin{cases} 4 - \lambda, & \lambda \geq 3 \\ -\frac{4}{27}(3 - \lambda)^3 + 4 - \lambda, & 0 \leq \lambda < 3. \end{cases} \)

\[ x = 0, +\infty, \quad \nabla_x h(x, \lambda) = 0 \Rightarrow 3x^2 - 2(3 - \lambda)x = 0 \]
\[ \Rightarrow x = 0 \text{ or } x = \frac{2}{3} (3 - \lambda) \]

iii. Conclude that \( d^* = 2 \) and the optimal \( \lambda = \frac{3}{2} \).

iv. Does strong duality hold?

\[ d^* = 2 = p^* \Rightarrow \text{YES strong duality!} \]

If you have a nonconvex problem you want to solve, try different equivalent specifications of your constraints.
2. Linear programming

Express the following problems as LPs.

(a) \( \min_{\mathbf{x} \in \mathbb{R}^k} \left[ \max_{i=1,...,k} x_i - \min_{j=1,...,k} x_j \right] \)

\( \text{s.t.} \quad A\mathbf{x} = \mathbf{b} \)

(b) \( \min \sum_{i=1}^{k} |x_i| \)

\( \text{s.t.} \quad A\mathbf{x} = \mathbf{b} \)