1. Complementary slackness
Consider the problem:

\[ p^* = \min_{x \in \mathbb{R}} x^2 \]
\[ \text{s.t. } x \geq 1, \ x \leq 2. \]

(a) Does Slater’s condition hold? Is the problem convex? Does strong duality hold?

(b) Find the Lagrangian \( L(x, \lambda_1, \lambda_2) \).

(c) Solve for \( x^*, \lambda_1^*, \lambda_2^* \) that satisfy KKT conditions.

(d) Can you spot a connection between the values of \( \lambda_1^*, \lambda_2^* \) in relation to whether the corresponding inequality constraints are strict or not at the optimal \( x^* \)?

(e) Find the dual function \( g(\lambda_1, \lambda_2) \) so that the dual problem is given by,

\[ d^* = \max_{\lambda_1, \lambda_2 \in \mathbb{R}^+} g(\lambda_1, \lambda_2). \] (1)

(f) Solve the dual problem in (1) for \( d^* \).

2. [Optional] Simple constrained optimization problem with duality
Consider the optimization problem

\[ \min_{x_1, x_2 \in \mathbb{R}} f(x_1, x_2) \]
\[ \text{subject to } 2x_1 + x_2 \geq 1 \]
\[ x_1 + 3x_2 \geq 1 \]
\[ x_1 \geq 0, \ x_2 \geq 0 \]

(a) Express the Lagrangian of the problem \( L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \)

Solve the following problems analytically and give the minimizing \( x_1^*, x_2^* \): Hint: Use duality if the problem is hard to solve. Use the graphs in Figure 2 to ”dualize” only some constraints:

(b) \( f(x_1, x_2) = x_1 + x_2 \)
(c) \( f(x_1, x_2) = -x_1 - x_2 \)
(d) \( f(x_1, x_2) = x_1 \)
(e) \( f(x_1, x_2) = \max\{x_1, x_2\} \)
(f) \( f(x_1, x_2) = x_1^2 + 9x_2^2 \)
Figure 1: Heatmap of 2(b), 2(c), 2(d), 2(e) and 2(f). $\vec{x}^\star = (\frac{2}{5}, \frac{1}{5})$. In red is the unfeasible points, then the level sets are shown with colors; blue points are points $(x_1, x_2)$ with the lowest value $f(x_1, x_2)$, red points are the ones with highest value.