1. A simple example of strong duality

Consider the following minimization problem, with $\epsilon \in \mathbb{R}$:

$$ p^* = \min_{x \in \mathbb{R}} x^2 $$

s.t. $x \geq \epsilon$

(a) Solve this optimization problem for $p^*$.

**Solution:**

$$ x^* = \max\{0, \epsilon\} $$

so $p^* = \begin{cases} 
0 & \text{if } \epsilon \leq 0 \\
\epsilon^2 & \text{otherwise}
\end{cases}$

(b) Write the Lagrangian function $\mathcal{L}(x, \lambda)$.

**Solution:**

$$ \mathcal{L}(x, \lambda) = x^2 + \lambda(\epsilon - x) $$

(c) Write the Lagrangian dual function $g(\lambda)$.

**Solution:**

$$ g(\lambda) = \min_{x} \mathcal{L}(x, \lambda) $$

$\mathcal{L}$ is convex and differentiable in $x$, so $x^*$ is such that $\nabla_x \mathcal{L}(x^*, \lambda) = 0$.

This gives that $x^* = \frac{\lambda}{2}$.

So $g(\lambda) = \frac{1}{4} \lambda^2 + \lambda(\epsilon - \frac{\lambda}{2}) = -\frac{1}{4} \lambda^2 + \lambda \epsilon$
(d) Solve the dual problem. Does strong duality hold?

**Solution:** \[ g(\lambda) = -\frac{1}{4}\lambda^2 + \lambda \epsilon = -\frac{1}{4}(\lambda - 2\epsilon)^2 + \epsilon^2. \]

\[ d^* = \max_{\lambda \geq 0} g(\lambda) = \begin{cases} 0 & \text{if } \epsilon \leq 0 \\ \epsilon^2 & \text{otherwise} \end{cases} \]

Here \( d^* = p^* \), strong duality holds.

(e) Give the value of the dual variable \( \lambda \) that maximizes the dual problem as a function of \( \epsilon \).

**Solution:** \( \lambda^* = \max\{0, 2\epsilon\} \)

2. Dual of an LP

Consider the general form of a linear program:

\[
\begin{align*}
\min_{\bar{x}} & \quad \bar{c}^T \bar{x} \\
\text{s.t.} & \quad A\bar{x} = \bar{b}
\end{align*}
\]

(a) Write the Lagrangian function \( \mathcal{L}(\bar{x}, \bar{\mu}) \).

**Solution:**

\[ \mathcal{L}(\bar{x}, \bar{\mu}) = \bar{c}^T \bar{x} + \bar{\mu}^T (A\bar{x} - \bar{b}) \]

(b) Write the Lagrangian dual function \( g(\bar{\mu}) \).

**Solution:**

\[ g(\bar{\mu}) = \inf_{\bar{x}} \mathcal{L}(\bar{x}, \bar{\mu}) = \begin{cases} -\bar{b}^T \bar{\mu} & \text{if } \bar{c} + A^T \bar{\mu} = \bar{0} \\ -\infty & \text{otherwise} \end{cases} \]
(c) Write the dual problem.

**Solution:** Substituting, we get

\[ d^* = \max_{\bar{\mu}} - \bar{b}^T \bar{\mu} \]

s.t. \( A^T \bar{\mu} + \bar{c} = \bar{0} \)