1. Simple constrained optimization problem with duality

Consider the optimization problem

\[
\begin{align*}
\min_{x_1, x_2} & \quad f(x_1, x_2) \\
\text{subject to} & \quad 2x_1 + x_2 \geq 1 \\
& \quad x_1 + 3x_2 \geq 1 \\
& \quad x_1 \geq 0, \ x_2 \geq 0
\end{align*}
\]

(a) Express the Lagragian of the problem \( \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \)

Solution:

\[
\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = f(x_1, x_2) + \lambda_1(-2x_1 - x_2 + 1) + \lambda_2(1 - x_1 - 3x_2) - \lambda_3 x_1 - \lambda_4 x_2
\]

(b) Show that \( \mathcal{L} \) is concave in \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\).

Solution: \(-\mathcal{L}\) is convex in \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) as a affine function of \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\). So \( \mathcal{L} \) is concave in \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\)

(c) Express the dual function of the problem, and show that it is concave.

Solution: \( g(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{x_1, x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \).

We can show that by showing that \(-g\) is convex.

\[
-g(\vec{\lambda}) = -\min_{x_1, x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)
\]

\[
= \max_{x_1, x_2} -\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)
\]

When \((x_1, x_2)\) is fixed, the function \(-\mathcal{L}\) is linear in \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\), therefore convex. Because the max of convex functions is convex, \(-g\) is convex. Therefore \(g\) is concave.

(d) Assume \(f\) is convex. Show that \(\mathcal{L}\) is convex in \((x_1, x_2)\).

Solution: \(\mathcal{L}\) is convex in \((x_1, x_2)\) because it is the sum of convex functions.

(e) Denoting \(\mathcal{X} = \{(x_1, x_2) \mid 2x_1 + x_2 \geq 1, \ x_1 + 3x_2 \geq 1, \ x_1 \geq 0, \ x_2 \geq 0\}\), show that

\[
\max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases}
\]

Solution: Let’s just do it for \(\lambda_4\):
\[
\max_{\lambda_1 \geq 0} L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \max_{\lambda_4 \geq 0} \left( f(x_1, x_2) + \lambda_1 (-2x_1 - x_2 + 1) + \lambda_2 (1 - x_1 - 3x_2) - \lambda_3 x_1 - \lambda_4 x_2 \right)
\]

\[
= f(x_1, x_2) + \lambda_1 (-2x_1 - x_2 + 1) + \lambda_2 (1 - x_1 - 3x_2) - \lambda_3 x_1 + \max_{\lambda_4 \geq 0} -\lambda_4 x_2
\]

\[
\max_{\lambda_4 \geq 0} -\lambda_4 x_2 = \begin{cases} 
0 & \text{if } x_2 \geq 0 \\
+\infty & \text{otherwise}
\end{cases}
\]

One can show the same results for \(\lambda_1, \lambda_2\) and \(\lambda_3\), resulting in:

\[
\max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} 
 f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\
+\infty & \text{otherwise}
\end{cases}
\]

(f) Conclude that \(\min_{(x_1, x_2) \in \mathcal{X}} \max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{(x_1, x_2) \in \mathcal{X}} f(x_1, x_2)
\]

**Solution:**

\[
\min_{x_1, x_2} \max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{x_1, x_2} f(x_1, x_2)
\]

(g) Assuming \(f\) is convex, formulate the first order condition on \(\mathcal{L}\) as a function of \(\nabla f\) and \(\lambda_1, \lambda_2, \lambda_3\) and \(\lambda_4\) to solve:

\[
\min_{x_1, x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)
\]

**Solution:**

\[
\nabla_{x_1, x_2} \mathcal{L}(x_1^*, x_2^*, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{bmatrix} 
-2\lambda_1 - \lambda_2 - \lambda_3 \\
-\lambda_1 - 3\lambda_2 - \lambda_4
\end{bmatrix}
\]

\[
= \nabla_{x_1, x_2} f(x_1^*, x_2^*) + \begin{bmatrix} 
-2\lambda_1 - \lambda_2 - \lambda_3 \\
-\lambda_1 - 3\lambda_2 - \lambda_4
\end{bmatrix}
\]

2. **Lagrangian Dual of a QP**

Consider the general form of a convex quadratic program, with \(Q > 0\):

\[
\min_{\bar{x}} \frac{1}{2} \bar{x}^\top Q \bar{x}
\]

\[
s.t. \quad A \bar{x} \leq \bar{b}
\]

(a) Write the Lagrangian function \(\mathcal{L}(\bar{x}, \bar{\lambda})\).

**Solution:**

\[
\mathcal{L}(\bar{x}, \bar{\lambda}) = \frac{1}{2} \bar{x}^\top Q \bar{x} + \bar{\lambda}^\top (A \bar{x} - \bar{b})
\]
(b) Write the Lagrangian dual function, \( g(\bar{\lambda}) \).

**Solution:**

\[
\begin{align*}
g(\bar{\lambda}) &= \inf_{\bar{x}} \mathcal{L}(\bar{x}, \bar{\lambda}) \\
&= \inf_{\bar{x}} \left( f_0(\bar{x}) + \sum_{i=1}^{n} \lambda_i f_i(\bar{x}) \right)
\end{align*}
\]

We can find this infimum by setting \( \nabla_{\bar{x}} \mathcal{L}(\bar{x}^*, \bar{\lambda}) = 0: \)

\[
Q\bar{x}^* + A^\top \bar{\lambda} = 0 \implies \bar{x}^* = -Q^{-1}A^\top \bar{\lambda}
\]

Substituting, we get

\[
g(\bar{\lambda}) = \mathcal{L}(\bar{x}^*, \bar{\lambda}) \\
&= \frac{1}{2} \bar{x}^\top A Q^{-\top} A^\top \bar{\lambda} - \bar{x}^\top A Q^{-1} A^\top \bar{\lambda} - \bar{\lambda}^\top \bar{b} \\
&= -\frac{1}{2} \bar{x}^\top A Q^{-1} A^\top \bar{\lambda} - \bar{\lambda}^\top \bar{b}
\]

(c) Show that the Lagrangian dual problem is convex by writing it in standard QP form. Is the Lagrangian dual problem convex in general?

**Solution:** The Lagrangian dual problem writes

\[
\max_{\bar{\lambda} \geq 0} g(\bar{\lambda}) = \max_{\bar{\lambda} \geq 0} \left( f_0(\bar{x}) + \sum_{i=1}^{n} \lambda_i f_i(\bar{x}) \right)
\]

the maximization of a concave function of \( \bar{\lambda} \) over the convex region given by the non-negative orthant \( \bar{\lambda} \geq 0 \). The dual problem is therefore convex.

While in this problem, the primal problem was convex, it turns out that the Lagrangian dual problem is a convex problem even when the primal is not. To see this, examine its general form:

\[
\max_{\bar{\lambda} \geq 0} \min_{\bar{x}} \mathcal{L}(\bar{x}, \bar{\lambda}) = \max_{\bar{\lambda} \geq 0} \min_{\bar{x}} \left( f_0(\bar{x}) + \sum_{i=1}^{n} \lambda_i f_i(\bar{x}) \right)
\]

This represents the pointwise minimum of affine functions of \( \bar{\lambda} \), which we know to be concave. The resulting maximization problem of a concave objective in \( \bar{\lambda} \) over the convex region \( \bar{\lambda} \geq 0 \) is then a convex optimization problem!