1. Change of Variables

(a) Suppose that $X$ has the **standard normal distribution**, that is, $X$ is a continuous random variable with density function

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

What is the density function of $\exp X$? (The answer is called the **lognormal distribution**.)

(b) Suppose that $X$ is a continuous random variable with density $f$. What is the density of $X^2$?

(c) What is the answer to the previous question when $X$ has the standard normal distribution? (This is known as the **chi-squared distribution**.)

**Solution:**

(a) We observe that for $x > 0$,

$$P(\exp X \leq x) = P(X \leq \ln x) = F(\ln x),$$

where $F$ is the CDF of the standard normal distribution. So,

$$f_{\exp X}(x) = \frac{d}{dx} P(\exp X \leq x) = \frac{d}{dx} P(X \leq \ln x) = F'(\ln x) \cdot \frac{1}{x} = \frac{f(\ln x)}{x} = \frac{1}{\sqrt{2\pi}x} \exp\left(-\frac{(\ln x)^2}{2}\right), \quad x > 0.$$

(b) We have $P(X^2 \leq x) = P(-\sqrt{x} \leq X \leq \sqrt{x}) = \int_{-\sqrt{x}}^{\sqrt{x}} f(x) \, dx$. So, the density of $X^2$ is

$$f_{X^2}(x) = \frac{1}{2\sqrt{x}} (f(-\sqrt{x}) + f(\sqrt{x})).$$

(c) When $X$ has the standard normal distribution,

$$f_{X^2}(x) = \frac{1}{2\sqrt{x}} \frac{1}{\sqrt{2\pi}} \left( \exp\left(-\frac{x}{2}\right) + \exp\left(-\frac{x}{2}\right) \right) = \frac{1}{\sqrt{2\pi}x} \exp\left(-\frac{x}{2}\right).$$

2. Revisiting Facts Using Transforms
(a) Let $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$ be independent. Calculate the MGF of $X + Y$ and use this to show that $X + Y \sim \text{Poisson}(\lambda + \mu)$.

(b) Calculate the MGF of $X \sim \text{Exponential}(\lambda)$ and use this to find all of the moments of $X$.

(c) Repeat the above part, but for $X \sim \mathcal{N}(0, 1)$.

**Solution:**

(a) The MGF of $X$ is

$$E[\exp(sX)] = \sum_{x \in \mathbb{N}} \exp(sx) \frac{\exp(-\lambda)\lambda^x}{x!} = \exp(-\lambda) \sum_{x \in \mathbb{N}} \frac{(\lambda \exp s)^x}{x!} = \exp(\lambda(\exp s - 1))$$

which converges for all $s \in \mathbb{R}$. The MGF of $X + Y$ is

$$E[\exp(s(X + Y))] = E[\exp(sX) \exp(sY)] = E[\exp(sX)] E[\exp(sY)]$$

$$= \exp(\lambda(\exp s - 1) + \mu(\exp s - 1))$$

$$= \exp((\lambda + \mu)(\exp s - 1))$$

which we recognize as the MGF of a Poisson$(\lambda + \mu)$ random variable. Observe that we used the independence of $X$ and $Y$ in the first line.

For this case, the fact that the MGF of $X + Y$ matches the MGF of the Poisson$(\lambda + \mu)$ distribution is enough to imply that $X + Y \sim \text{Poisson}(\lambda + \mu)$.

In general, it is not easy to argue that the MGF uniquely determines the probability distribution (this also requires a few assumptions on the MGF itself), but we will not worry about these issues in this course.

(b) We calculate

$$M_X(s) = \int_0^\infty \exp(sx)\lambda \exp(-\lambda x) \, dx = \lambda \int_0^\infty \exp(-(\lambda - s)x) \, dx$$

$$= \frac{\lambda}{\lambda - s},$$

which converges for $s < \lambda$. Expanding $M_X$ as a geometric series,

$$M_X(s) = \frac{1}{1 - s/\lambda} = \sum_{k \in \mathbb{N}} \left(\frac{s}{\lambda}\right)^k$$

as long as $|s| < \lambda$. Compare this last expression with

$$E[\exp(sX)] = E\left[\sum_{k \in \mathbb{N}} \frac{(sX)^k}{k!}\right] = \sum_{k \in \mathbb{N}} \frac{s^k E[X^k]}{k!}$$

and matching terms, we can argue that $E[X^k] = k! / \lambda^k$. 
(c) The MGF of the standard Gaussian is

\[ M_X(s) = \int_{-\infty}^{\infty} \exp(sx) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \]

\[ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x^2 - 2sx)\right) dx \]

\[ = \left( \exp\left(\frac{s^2}{2}\right) \right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x^2 - 2sx + s^2)\right) dx \]

\[ = \left( \exp\left(\frac{s^2}{2}\right) \right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\left(\frac{x-s}{\sqrt{2}}\right)^2\right) dx = \exp\left(\frac{s^2}{2}\right). \]

Expanding as a power series,

\[ M_X(s) = \sum_{k \in \mathbb{N}} \frac{s^{2k}}{2^k k!} \]

so by comparing terms as before, we see that \( \mathbb{E}[X^k] = 0 \) if \( k \) is odd and \( \mathbb{E}[X^k] = k!/\left[2^{k/2}(k/2)!\right] = (k-1)!! \) if \( k \) is even.

3. Exponential Bounds

Let \( X \sim \text{Exponential}(\lambda) \). For \( x > \lambda^{-1} \), calculate bounds on \( \mathbb{P}(X \geq x) \) using Markov’s Inequality, Chebyshev’s Inequality, and the Chernoff Bound.

**Solution:**

Since \( \mathbb{E}[X] = \lambda^{-1} \), Markov’s Inequality gives

\[ \mathbb{P}(X \geq x) \leq \frac{\mathbb{E}[X]}{x} = \frac{1}{\lambda x}. \]

and from \( \text{var} \ X = \lambda^{-2} \), Chebyshev’s Inequality gives

\[ \mathbb{P}(X \geq x) = \mathbb{P}(X - \lambda^{-1} \geq x - \lambda^{-1}) \leq \mathbb{P}(|X - \lambda^{-1}| \geq x - \lambda^{-1}) \]

\[ \leq \frac{\text{var} \ X}{(x - \lambda^{-1})^2} = \frac{1}{(\lambda x - 1)^2}. \]

For the Chernoff Bound, for any \( s > 0 \),

\[ \mathbb{P}(X \geq x) = \mathbb{P}(\exp(sX) \geq \exp(sx)) \leq \frac{M_X(s)}{\exp(sx)} = \frac{\lambda}{(\lambda - s) \exp(sx)}. \]

We wish to optimize this bound over \( s > 0 \). It suffices to maximize the denominator \( (\lambda - s) \exp(sx) \). Taking derivatives,

\[ -\exp(sx) + x(\lambda - s) \exp(sx) = 0, \]

so \( 1 = x(\lambda - s) \), that is, \( s = \lambda - x^{-1} \). Thus,

\[ \mathbb{P}(X \geq x) \leq \frac{\lambda}{(\lambda - (\lambda - x^{-1})) \exp((\lambda - x^{-1})x)} = \frac{\lambda}{x^{-1} \exp(\lambda x - 1)} \]

\[ = \lambda x \exp\left(-\left(\lambda x - 1\right)\right). \]

Observe that the Chernoff Bound is the only one which decreases exponentially with \( x \), which is the true behavior: \( \mathbb{P}(X \geq x) = \exp(-\lambda x) \).
4. First Time to Decrease

Let \(X_1, X_2, \ldots, X_n, \ldots\) be a sequence of independent and identically distributed (i.i.d.) continuous random variables with common PDF \(f\).

(a) Argue that \(\mathbb{P}(X_i = X_j) = 0\) for \(i \neq j\).

(b) Calculate \(\mathbb{P}(X_1 \leq X_2 \leq \cdots \leq X_{n-1})\).

(c) Let \(N\) be a random variable which is equal to the first time that the sequence of the random variables will decrease, i.e.

\[ N = \min\{n \in \mathbb{Z}_{\geq 2} \mid X_{n-1} > X_n\} \]

Calculate \(\mathbb{E}[N]\).

Solution:

(a) \[
\mathbb{P}(X_i = X_j) = \int_{\{(x_i, x_j) \in \mathbb{R}^2 \mid x_i = x_j\}} f(x_i) f(x_j) \, dx_i \, dx_j
\]

\[= \int_{-\infty}^{\infty} f(x_j) \int_{x_j}^{\infty} f(x_i) \, dx_i \, dx_j = 0.\]

(b) Because the random variables are i.i.d. we have that for any permutation \(\sigma\) of \(\{1, 2, \ldots, n-1\}\),

\[\mathbb{P}(X_1 \leq X_2 \leq \cdots \leq X_{n-1}) = \mathbb{P}(X_{\sigma(1)} \leq X_{\sigma(2)} \leq \cdots \leq X_{\sigma(n-1)}) .\]

In addition from the previous part we know that equality between the continuous i.i.d. random variables will occur with probability 0 and so

\[\mathbb{P}(X_{\sigma(1)} \leq X_{\sigma(2)} \leq \cdots \leq X_{\sigma(n-1)}) = \mathbb{P}(X_{\sigma(1)} < X_{\sigma(2)} < \cdots < X_{\sigma(n-1)}).\]

Since there are \((n-1)!\) possible permutations, and we know that

\[\sum_{\sigma} \mathbb{P}(X_{\sigma(1)} \leq X_{\sigma(2)} \leq \cdots \leq X_{\sigma(n-1)})
\]

\[= \sum_{\sigma} \mathbb{P}(X_{\sigma(1)} < X_{\sigma(2)} < \cdots < X_{\sigma(n-1)}) = 1,\]

we can conclude that

\[\mathbb{P}(X_1 \leq X_2 \leq \cdots \leq X_{n-1}) = \frac{1}{(n-1)!}.\]

(c) Observe that \(\mathbb{P}(N \geq 1) = 1\) and for \(n \geq 2\) we have that \(\mathbb{P}(N \geq n) = \mathbb{P}(X_1 \leq X_2 \leq \cdots \leq X_{n-1}) = \frac{1}{(n-1)!}\). Therefore using the tail sum formula for the expectation we have that

\[\mathbb{E}[N] = \sum_{n=1}^{\infty} \mathbb{P}(N \geq n) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e.\]