University of California - Berkeley
Department of Electrical Engineering & Computer Sciences
EE126 Probability and Random Processes
(Spring 2012)

Homework 8
Due Tuesday, April 17th

1. (a) Given the information $\mathbf{E}[X] = 7$ and $\text{var}(X) = 9$, use the Chebyshev inequality to find a lower bound for $\mathbf{P}(4 \leq X \leq 10)$.
   (b) Find the smallest and largest possible values of $\mathbf{P}(4 < X < 10)$, given the mean and variance information from part (a).

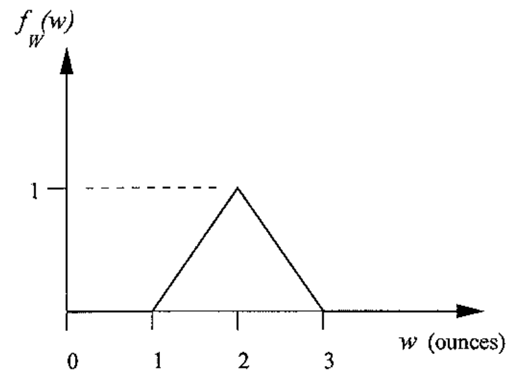
2. Investigate whether the Chebyshev inequality is tight. That is, for every μ, $\sigma \geq 0$, and $c \geq \sigma$, does there exist a random variable X with mean μ and standard deviation σ such that

$$\mathbf{P}(|X - \mu| \geq c) = \frac{\sigma^2}{c^2}?$$

3. Define X as the height in meters of a randomly selected Canadian, where the selection probability is equal for each Canadian, and denote $\mathbf{E}[X]$ by h. Bo is interested in estimating h. Because he is sure that no Canadian is taller than 3 meters, Bo decides to use 1.5 meters as a conservative (large) value for the standard deviation of X. To estimate h, Bo averages the heights of n Canadians that he selects at random; he denotes this quantity by H.
   (a) In terms of h and Bo's 1.5 meter bound for the standard deviation of X, determine the expectation and standard deviation for H.
   (b) Help Bo by calculating a minimum value of n (with $n > 0$) such that the standard deviation of Bo's estimator, H, will be less than 0.01 meters.
   (c) Say Bo would like to be 99% sure that his estimate is within 5 centimeters of the true average height of Canadians. Using the Chebyshev inequality, calculate the minimum value of n that will make Bo happy.
   (d) If we agree that no Canadians are taller than three meters, why is it correct to use 1.5 meters as an upper bound on the standard deviation for X, the height of any Canadian selected at random?

4. Let $X_1, X_2, \ldots$ be independent, identically distributed, continuous random variables with $\mathbf{E}[X] = 2$ and $\text{var}(X) = 9$. Define $Y_i = (0.5)^i X_i$, $i = 1, 2, \ldots$. Also define T_n and A_n to be the sum and the average, respectively, of the terms $Y_1, Y_2, \ldots, Y_n$.
   (a) Is Y_n convergent in probability? If so, to what value? Explain.
   (b) Is T_n convergent in probability? If so, to what value? Explain.
   (c) Is A_n convergent in probability? If so, to what value? Explain.

5. The weight of a Pernotti Parabolic Pretzel, W, is a continuous random variable described by the probability density function



$$f_W(w) = \begin{cases} 0 & w \le 1 \\ w-1 & 1 \le w \le 2 \\ 3-w & 2 \le w \le 3 \\ 0 & 3 \le w \end{cases}$$

What is the probability that 102 pretzels (with independent weights) will have a total weight of more than 200 ounces? Find an approximate answer.

6. Your friend challenges you to a coin flipping contest. However you know this friend to be of questionable moral character. In fact you know that he usually caries a weighted coin that comes up heads with probability 0.55, along with a fair coin. You demand that he flip one coin 1000 times, and if it comes up heads more than 525 times, then you will play using the other coin. Assuming he uses the fair coin, find an approximation for the probability that you will think it is the biased coin.

7. On any given flight, an airline's goal is to have the plane be as full as possible, without overbooking. If, on average, 10cancel their tickets, all independently of each other, what is the probability that a particular flight with maximum capacity 300 people will be overbooked if the airline sells 320 tickets? Find an approximate answer.

8. The length in meters, X, of each section of pipe we obtain is an independent discrete random variable with PMF:

$$p_X(k) = (\frac{1}{2})^k \qquad k = 1, 2, \ldots$$

$$E[X] = 2 \qquad \mathrm{var}(X) = 2$$

(a) Suppose we obtain 400 sections of pipe. Determine the value of a bound, w, such that the total length of the sections we obtain will be greater than w with a probability of approximately 0.841.

(b) Determine n, the number of sections of pipe needed such that the probability we obtain at least 200 meters of pipe is approximately 0.841.