UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Problem Set 3

Spring 2006

Issued: Thursday, February 2, 2006 **Due:** Thursday, February 9, 2006

Reading: Berstsekas & Tsitsiklis, §1.5, §1.6. and §2.1

Problem 3.1
Consider a communication channel model where the transmitter can send a binary input of either "0" or "1", and where the output of the channel is also binary. Due to noise, the probability of receiving a "0" given that a "0" was transmitted is equal to $(1-\epsilon)$, and that of receiving a "1" given that "1" was transmitted is equal $(1-\delta)$. We assume that ϵ and δ are less than half.

(a) What is the probability of receiving "1" when "0" is transmitted? What is the probability of receiving "0" when "1" is transmitted?

(b) Assume that the transmitter is three times more likely to transmit "1" than "0". What is the probability of reception error?

(c) Instead of directly transmitting a bit, the transmitter decides to "encode" his message by repeatedly transmitting $(2n+1)$ times the bit he needs to communicate to the receiver. Accordingly, the receiver uses a majority rule to decide what bit was transmitted (among $(2n+1)$ channel outputs, if more "1"'s are received than "0"'s, then the receiver decides that the transmitter message was "1" and vice versa). Using this strategy, and assuming that channel errors are independent, compute the message probability of error when the assumption of part (b) still holds.

(d) When $\delta = \epsilon = 0.1$, how many times does the transmitter need to repeat his message in order to guarantee a probability of error less than 10^{-6}?

Problem 3.2
We are told that events A and B are independent. In addition, events A and C are independent. Is it true that A is independent of $B \cup C$? Provide a proof or counterexample to support your answer.

Problem 3.3
The Miami Heat and LA Lakers are going to play a series consisting of n games, where n is odd. The team that wins the most games wins the series. At any given game, the Lakers have a probability p of winning, independently of any other game. Assume they will play all n games.

(a) Find values of p for which the Lakers, who would really like to win the series, would rather play a 5 game series instead of a 3 game series.

(b) Find values of p for which the Lakers would prefer $n = 2k + 1$ instead of $n = 2k - 1$, for every integer $k > 0$.

Problem 3.4

George and his friend Bob love going to a strange racetrack, where only two horses—Heads and Tails—ever race. Heads wins with probability p, and Tails with probability $(1 - p)$. Bob and George are also somewhat peculiar, because they always insist on betting on Heads, and they also never bet at the same time. Bob will let George bet on Heads again and again until Heads loses, at which point Bob will bet on Heads again and again until he loses, and so on. Now suppose that George and Bob have a private bet amongst themselves: if George wins n races before Bob can win m races, then George wins, and otherwise Bob wins. For non-negative integers i and j, let $P_{i,j}$ be the probability that George, who always starts the betting game, wins i times before his friend Bob wins j times. Show that:

$$P_{n,m} = pP_{n-1,m} + (1 - p)(1 - P_{m,n}).$$

Problem 3.5

Alice plays with Bob the following game. First Alice randomly chooses 4 cards out of a 52-card deck, memorizes them, and places them back into the deck. Then Bob randomly chooses 8 cards out of the same deck. Alice wins if Bob's cards include all cards selected by her. What is the probability of this happening?

Problem 3.6

Twenty five black pebbles are arranged in five rows of five pebbles each. We choose five of these pebbles at random and color them red.

(a) What is the probability that all the red pebbles lie in different rows?

(b) What is the probability that all the red pebbles lie in different rows and in different columns?

Problem 3.7

An urn contains M balls, of which M_1 are white. A set of n balls is obtained either by sampling balls with replacement (each ball is replaced in the urn after it is drawn), or without replacement (a ball once drawn remains out of the urn). Let B_j be the event that the ball selected at the j^{th} step is white, and A_k the event that a sample of size n contains exactly k white balls. Show that

$$\mathbb{P}(B_j | A_k) = k/n,$$

both for sampling with replacement and for sampling without replacement.