UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probablity and Random Processes

Problem Set 11

Spring 2006

Issued: Friday, April 28, 2006 **Due:** Friday, May 5, 2006

Reading: Bertsekas & Tsitsiklis, §6.1–6.4; Chapter 7

Problem 11.1

Sam and Pat are playing foosball. When they begin, the score is 0-0. To make things interesting, if the score ever becomes tied, it is instantly reset to 0-0. Starting from any score, the probability that Sam gets the next point is $\frac{1}{3}$.

(a) Suppose the game stops when one player's score reaches 2.

(i) Draw an appropriate Markov chain that describes the game.

(ii) Identify all transient, recurrent, and periodic states.

(iii) Find $\mathbb{P}$(Pat wins).

(b) Now suppose instead that the game stops when a total of 3 points have been scored (note that this stopping condition does not explicitly depend on the score). The player with the most points when the game ends wins. Draw an appropriate Markov chain that describes the game.

Problem 11.2

Imagine that on a particular roulette wheel, $\mathbb{P}(\text{WIN}) = \frac{18}{37}$. Suppose that you play 100 games, and are interested in finding the probability that you win at least half of them.

(a) Write an exact summation for this probability, but do not try to evaluate it explicitly.

(b) Use the central limit theorem to derive a suitable approximation to this probability, and compute the value of your approximation.

Problem 11.3

The test scores of 900 students had the following sample statistics:

Mean: 83 Variance: 36

Use Chebyshev's inequality to bound the probability that a randomly selected student received a test score between 71 and 95 inclusive. Is it likely that at least 600 students scored between 71 and 95 inclusive? Why or why not?

Problem 11.4

Discrete random variable X is equal to 0 with probability 0.5 and otherwise takes on values -2 and 2 with equal probabilities. Let X_1, X_2, ... , be independent identically distributed random variables with the same distribution as X. For each of the following sequences, determine whether each converges in probability and, if so, the limit to which it converges. Justify your answer.

(a) $Y_n = \max(X_1, X_2, ..., X_n)$

(b) $T_n = X_1 + X_2 + ... + X_n$

(c) $A_n = \frac{X_1+X_2+...+X_n}{n}$

Problem 11.5
Samantha the Gambler likes to play poker. Being a good player, the amount that she wins on a given night i can be modeled as a Gaussian X_i with mean $\mu > 0$ and variance $\sigma^2 = 1$. Let Y be the total amount of money that Samantha wins after n nights of playing poker.

(a) Write down an integral expression for the exact probability that Samantha wins more than α dollars after n nights of gambling. Do *NOT* try to evaluate this integral.

(b) Obtain an upper bound on the probability that Samantha wins more than α dollars after n nights of gambling using the Chernoff bound. Choose a value for s which yields the best (i.e., smallest) upper bound. Compute the value of this smallest upper bound if $\alpha = 90$, $n = 100$ and $\mu = 0.5$.

Notice that the bound you obtain is much simpler to use than your integral expression from (a).

Problem 11.6
Wombats and dingos arrive in a Poisson manner to a particular water hole in the Australian Outback. The arrival rates of wombats and dingos are 2 and 4 per hour, respectively. Each animal will stay and drink until the next animal arrives to take over. No other animals visit the water hole.

(a) What is the expected number of animals (wombats or dingos) that visit the water hole in a 24-hour period?

(b) Given that a wombat is currently drinking, what is the probability that the next animal to visit is a dingo?

Crocodile Dundee arrives at the water hole at a random time and leaves the water hole immediately after the $900th$ animal he sees departing the water hole.

(c) How long, on average, will Crocodile Dundee have to wait to see a dingo?

(d) Consider the first dingo that Crocodile Dundee sees. How long, on average, does this dingo spend at the water hole?

(e) What does Chebyshev's inequality tell you about the probability that Crocodile Dundee stays at the water hole for between 140 and 160 hours?

(f) Using the approximation provided by the Central Limit Theorem, what is the probability that Crocodile Dundee stays at the water hole for between 140 and 160 hours?