UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Problem Set 10
Spring 2006

Issued: Friday, April 21, 2006 **Due:** Friday, April 28, 2006

Reading: Bertsekas & Tsitsiklis, Chapter 6

Problem 10.1
John's office is in a building that has 5 doors. Due to personal peculiarity, John refuses to use the same door twice in a row. In fact, he will choose the door to the left or the right of the last door he used, with probability p and $1-p$ respectively, and independently of what he has done in the past. For example, if he just chose door 5, there is a probability p that he'll choose door 4, and a probability $1-p$ that he'll choose door 1 next.

(a) Argue that the above process is a Markov chain.

(b) Find the transition probability matrix.

(c) Find the steady state probabilities.

Problem 10.2

Two thimbles (like tiny cups) are under a dripping roof. At the end of each second, thimble A receives 1 drop of water with probability 1, and thimble B receives 1 drop with probability 2/3 and 0 drops otherwise.

By a complicated automatic mechanism, right before a 4th drop lands in thimble A, both thimbles are emptied. While the thimbles are being emptied, they miss catching the drops that would have otherwise landed inside the thimbles.

(a) Set up a Markov chain model: identify the states, draw the state transition diagram, and indicate the transition probabilities.

(b) If both thimbles were empty when you started watching, what is the probability that both thimbles contain exactly 1 drop after exactly 10,001 seconds?

Problem 10.3
For each of the following definitions of state X_k at time k ($k = 1, 2, \ldots$), determine whether the Markov property is satisfied and, when it is, specify the transition probabilities p_{ij}:

(a) A six-sided die is rolled repeatedly.

(i) Let X_k denote the largest number rolled in the first k rolls.
(ii) Let X_k denote the number of sixes in the first k rolls.
(iii) At time k, let X_k be the number of rolls since the most recent six.

(b) Let Y_k be the state of some discrete-time Markov process at time k (i.e., it is known Y_k satisfies the Markov property) with known transition probabilities q_{ij}.

   (i) For a fixed integer $r > 0$, let $X_k = Y_{r+k}$.

   (ii) Let $X_k = Y_{2k}$.

   (iii) Let $X_k = (Y_k, Y_{k+1})$; that is, the state X_k is defined by the sequence of state *pairs* in a given Markov process.

Problem 10.4

A telephone company establishes a direct connection between two cities expecting Poisson traffic with rate c calls/minute. The durations of calls are independent and exponentially distributed with mean $\frac{1}{d}$ minutes. Interarrival times are independent of call durations. The system can handle up to n calls, meaning that any calls that are attempted when the system has n calls on line will be blocked. Each time a call comes in (regardless of whether or not it is blocked), a red light flashes. Each time a call (that was not blocked) ends, a green light flashes. Assume we start observing the system long after the process has started.

(a) Given that the number of calls on the system just changed, find the PMF for the number of calls on the system immediately after the change.

(b) Given that the number of calls on the system just changed, find the probability the number of calls decreased.

(c) Given that a light just flashed, find the PMF for the number of calls on the system immediately after the light flashed.

(d) Given that a light just flashed, find the probability the light was green.

(e) Given that the red light just flashed, fine the PMF for the number of calls on the system immediately after the light flashed.

Problem 10.5

A discrete-time Markov chain is known to be a birth-death process with three states—the probabilities $b_0 = b_1 = d_1 = 0.2$ but d_2 is unspecified. Let X_k denote the state at time $k = 0, 1, \ldots$. For each of the following statements, *either* determine the possible value(s) of probability d_2 *or* explain why no such value exists:

(a) Upon entering state 2, the expected time it takes to first exit state 2 is equal to 100.

(b) The probability $\mathbf{P}(X_2 = 2 \mid X_0 = 2)$ is equal to 0.28.

(c) Given no self-transitions ever occur, the probability $\mathbf{P}(X_4 = 0 \mid X_0 = 0)$ is equal to 0.5.

(d) The steady-state probability associated with state 1 is equal to 0.25.

(e) The long-term expected frequency of deaths is equal to 0.3.

(f) State 2 is an absorbing state.

(g) The mean recurrence time of state 2 is equal to 6.