UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Instructions and review for the final exam
Spring 2006

Exam format

- The second midterm will be held on Wednesday May 17; CHECK the final exam schedule. You are permitted to bring a calculator, and three sheets of hand-written notes on $8.5 \times 11''$ paper.
- Questions on the exam can be based on any material from Chapters 1 through to the END of chapter 7, as discussed in lectures #1 through #30, covered in homeworks #1 through #11, and all discussion sections.

Review problems

Problem 9.1

Let X and Y be independent random variables that are uniformly distributed on $[0, 1]$.

(a) Find the mean and variance of $X - 2Y$.

(b) Let A be the event $\{X \leq Y\}$. Find the conditional PDF of X given that A occurred. (A fully labeled sketch will suffice.)

Problem 9.2

Male and female patients arrive at an emergency room according to independent Poisson processes, with each process having a rate of 3 per hour. Let $M_{t,t'}$ be the number of male arrivals between t and t'. Let $F_{t,t'}$ be the number of female arrivals between time t and t'.

(a) Write down the PMF of $M_{3,5} + F_{4,6}$.

(b) Calculate the variance of $M_{3,5} + M_{4,6}$.

(c) What is the expectation of the arrival time of the *last* patient to arrive *before* 4 p.m.?

(d) Starting from a particular time, what is the expected time until there is an arrival of at least one male and at least one female patient?

Problem 9.3

Consider a factory that produces $X_n \geq 0$ gadgets on day n. The X_n are independent identically distributed random variables with mean 5, variance 9, $\mathbb{E}[X_n^3] = 412$, and $\mathbb{E}[X_n^4] < \infty$. Furthermore, we are told that $\mathbb{P}(X_n = 0) > 0$.

(a) Find an approximation to the probability that the total number of gadgets produced in 100 days is less than 440.

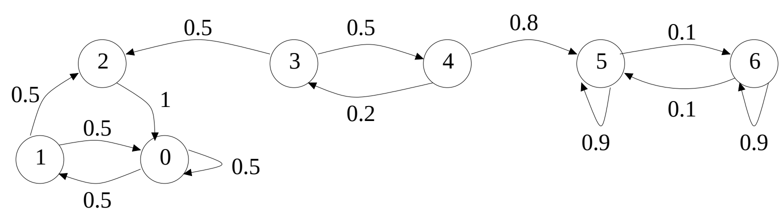
(b) Find (approximately) the largest value of n such that $\mathbb{P}(X_1+\cdots+X_n \geq 200+5n) \leq 0.05$.

(c) For each definition of Z_n given below, state whether the sequence Z_n converges in probability. (Answer yes or no, only.)

   (a) $Z_n = \frac{X_1+\cdots+X_n}{n}$.
   (b) $Z_n = \frac{X_1+\cdots+X_n-5n}{\sqrt{n}}$.
   (c) $Z_n = \frac{X_1^2+\cdots+X_n^2}{n}$.
   (d) $Z_n = X_1X_2\cdots X_n$.

Problem 9.4

Consider the Markov chain specified by the following state transition diagram.



Let X_n be the value of the state at time n.

(a) For all states $i = 0, 1, \ldots, 6$, find μ_i the expected time to absorption starting from state i, (i.e. entering a recurrent state.)

(b) Find $\lim_{n\to\infty} \mathbb{P}(X_n = 2|X_0 = 3)$.

Problem 9.5
The random variable X is distributed as a binomial random variable with parameters $n > 1$ and $0 < p < 1$. The pair of random variables (Y, Z) takes values on the set

$$\{(0,1), (1,0), (0,-1), (-1,0)\}$$

with equal probability if $X \leq n/2$. Similarly, (Y, Z) takes values on the set

$$\{(1,1), (1,-1), (-1,1), (-1,-1)\}$$

with equal probability if $X > n/2$.

(a) Are X and Y independent?

(b) Are Y and Z independent?

(c) Conditioned on X being even, are Y and Z independent?

(d) Conditioned on $Y = 1$, are X and Z independent?

(e) Conditioned on $Z = 0$, are X and Y independent?

Problem 9.6

Consider $N+1$ independent Poisson arrival processes, such that the i^{th} process has arrival rate $\lambda_i = i\lambda$, for $i = 1, 2, \ldots, (N + 1)$. Let N be a binomial random variable, independent of all the Poisson arrival processes; $\mathbb{E}[N] = \mu$ and $\text{var}(N) = v$.

(a) Assume that N represents the number of successes in n independent Bernoulli trials, each with a probability p of success. Find n and p in terms of μ and v.

(b) Find, in terms of λ, μ and v, the mean of the number of total arrivals from the sum of the processes, in a time interval of length t.

Problem 9.7

Let $X_1, X_2, \ldots$ be an i.i.d. sequence of normal random variables with mean μ and variance σ^2. Further, let $Y_n = \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}}$ for $n = 1, 2, \ldots$.

(a) Using the Chebyshev inequality, give an upper bound for $\mathbb{P}(|Y_n - \mathbb{E}[Y_n]| \geq \epsilon)$.

(b) Evaluate $\mathbb{P}(|Y_n - \mathbb{E}[Y_n]| \geq \epsilon)$ exactly in terms of Φ, the CDF of the standard normal random variable.

(c) Hence compute $\mathbb{P}(|Y_n - \mathbb{E}[Y_n]| \geq \epsilon)$ and its Chebyshev upper bound for $\epsilon/\sigma = 0.5$, $\epsilon/\sigma = 1.0$, and $\epsilon/\sigma = 2.0$.

(d) For $n > k$, find the linear least squares estimate of Y_n given $Y_k = y$ and its mean squared error.

Problem 9.8

The BART is broken, so that only two kinds of vehicles go from Berkeley to San Francisco: taxis and buses. The interarrival time of taxis, in minutes, is an independent exponential random variable with parameter λ_1, i.e. its PDF is $f_{I_T}(t) = \lambda_1 e^{-\lambda_1 t}$ for $t \geq 0$, while the interarrival time of buses, in minutes, is an independent exponential random variable with parameter λ_2, i.e., its PDF is $f_{I_B}(t) = \lambda_2 e^{-\lambda_2 t}$ for $t \geq 0$.

Suppose Joe and Harry arrive at Berkeley at 7:00 a.m.

(a) What is the average time before they see the first vehicle?

(b) What is the probability that the first vehicle they see is a bus, and what is the probability that the first vehicle they see is a taxi?

In a taxi, the travel time to San Francisco, in minutes, is an independent exponential random variable with parameter μ_1, i.e., its PDF is $f_{D_T}(t) = \mu_1 e^{-\mu_1 t}$ for $t \geq 0$. On the other hand, in a bus, the travel time to San Francisco, in minutes, is an an independent exponential random variable with parameter μ_2, i.e., its PDF is $f_{D_B}(t) = \mu_2 e^{-\mu_2 t}$ for $t \geq 0$.

(c) Suppose Joe and Harry arrive at downtown Berkeley at 7:00 a.m., take the first vehicle that passes, and arrive in San Francisco X minutes later. Find the transform of X.

(d) Suppose that a taxi and a bus arrive simultaneously, and Joe takes the taxi while Harry takes the bus. Let Y be the number of minutes from their departure from Berkeley till they meet again in San Francisco. Find $\mathbb{E}[Y]$.

There are, in fact, two different kinds of buses: fast buses and slow buses. For any bus that arrives at Berkeley downtown, it is a fast bus with probability p, and it is a slow bus with probability $1 - p$. Whether a bus is fast or slow is independent of everything else.

(e) If they stay at Berkeley for l minutes, how many fast buses will they see on average?

(f) If they stay indefinitely, what is the probability that they will see k fast buses before they see k slow buses?

Problem 9.9

A hungry mouse is trapped in a cage with three doors. At each "turn", the mouse gets to open one of the three doors and eat a piece of cheese behind the door if one is present. Each door is chosen with equal probability on each turn, regardless of whether a piece of cheese was found on the previous turn. If no cheese was found on the previous turn, there is a probability of $3/4$ that cheese will be found behind each door on the current turn. If cheese was found on the previous turn and the same door is chosen on the current turn, then there is a probability of 0 that cheese will be found; whilst if cheese was found on the previous turn and a different door is chosen on the current turn, then there is a probability of 1 that cheese will be found.

(a) If you observe the mouse's behavior over 1000 turns, in approximately what fraction of turns do you expect the mouse to eat a piece of cheese?

(b) Suppose no cheese was found on the previous turn. What is the expected number of turns before the mouse eats a piece of cheese?

(c) Suppose no cheese was found on the previous turn. What is the expected number of turns before the mouse eats n pieces of cheese?

(d) Suppose cheese was found on the previous turn. Using the Central Limit Theorem, approximate the probability that the number of turns before the mouse eats 100 pieces of cheese exceeds 152.

(e) You look into the cage and observe the mouse eating a piece of cheese from behind door number 1. What is the probability that, if you observe the mouse three turns later, it will again be eating a piece of cheese from behind door number 1?