UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probablity and Random Processes

Solutions 9
Spring 2006

Problem 9.1
[Midterm 2 Fa02]

(a) A continuous random variable U has a pdf of the form:

$$f_U(u) = ae^{bu^2+cu}, \quad -\infty < u < +\infty$$

where a, b, c are constants. a and b are non-zero.

i) Can you say anything definitive about the signs of a, b and c?

ii) Does U have to be Gaussian? If so, express its mean and variance in terms of a, b and c in the simplest way. If not, give an example of U with pdf of the above form but is not Gaussian.

(b) Let $X \sim N(0; v^2)$, find the moment generating function of X, $M_X(s)$.

(c) Let $X \sim N(0; v^2)$, $Z \sim N(0; \sigma^2)$ and we have a noisy observation $Y = X + Z$. The RVs X and Z are independent. Find from first principles the MMSE estimate of X given Y and the resulting minimum mean square error. (*Hint:* you may find your answer to part (a)(ii) useful in simplifying the calculations.).

Solution: See the handwritten solutions in the previous discussion section.

Problem 9.2
Let X be a Bernoulli random variable with parameter p. A random variable Z is a mixture of Gaussian random variables and is defined as

$$Z = XY_1 + (1 - X)Y_2$$

where Y_1 is $N(-1, \sigma^2)$ and Y_2 is $N(1, \sigma^2)$.

We have access to a noisy observation of Z, which we denote as V. V is given by Z corrupted by additive gaussian noise, i.e.

$$V = Z + W$$

where $\text{Var}(W) = \sigma_W^2$. Given V, find the estimator of Z that minimizes the mean squared error.

Solution: The estimator that minimizes the mean squared error is $E[Z|V]$. The expectation can be computed by further conditioning over X:

$$E[Z|V] = E[Z|V, X = 1]P(X = 1|V) + E[Z|V, X = 0]P(X = 0|V)$$

Given $X = 1$ we have that $Z = Y_1$ and $E[Z|V, X = 1] = E[Y_1|V]$. Since Y_1 and V are both gaussian, the conditional expectation is going to be a linear function of V:

$$E[Z|V, X = 1] = E[Y_1|V] = E[Y_1] + \rho\frac{\sigma_Z}{\sigma_V}(V - E[V])$$

and since $E[Y_1] = E[V] = -1$,

$$E[Z|V, X = 1] = E[Y_1] + \rho\frac{\sigma_{Y_1}}{\sigma_V}(V - E[V]) = -1 + \frac{\sigma^2}{\sigma^2 + \sigma_W^2}(V + 1)$$

Similarly, given $X = 0$ we have that $Z = Y_2$ and

$$E[Z|V, X = 0] = E[Y_2] + \rho\frac{\sigma_{Y_2}}{\sigma_V}(V - E[V]) = 1 + \frac{\sigma^2}{\sigma^2 + \sigma_W^2}(V - 1)$$

Next, we need to compute $q_V = P(X = 1|V) = 1 - P(X = 0|V)$. From Bayes' rule we have that:

$$\begin{aligned} q_V = P(X = 1|V) &= \frac{f_{V|X=1}P(X = 1)}{f_{V|X=1}P(X = 1) + f_{V|X=0}P(X = 0)} \\ &= \frac{\mathcal{N}(-1, \sigma^2 + \sigma_W^2)p}{\mathcal{N}(-1, \sigma^2 + \sigma_W^2)p + \mathcal{N}(1, \sigma^2 + \sigma_W^2)(1 - p)} \end{aligned}$$

So, for each value of V we can compute the number q_V. Finally, we have

$$E[Z|V] = q_V\left[-1 + \frac{\sigma^2}{\sigma^2 + \sigma_W^2}(V + 1)\right] + (1 - q_V)\left[1 + \frac{\sigma^2}{\sigma^2 + \sigma_W^2}(V - 1)\right]$$

Problem 9.3

Assume that X and Y are i.i.d. $N(0; 1)$. Calculate $E[(X + Y)^4|X - Y]$.

Solution: $X + Y$ and $X - Y$ are linear function of independent Gaussian random variables, so they are jointly Gaussian random variables. We can verify that they are uncorrelated

$$E[(X + Y)(X - Y)] = E[X^2] - E[Y^2] = 1 - 1 = 0.$$

It follows that $X + Y$ and $X - Y$ are independent because they are jointly Gaussian and uncorrelated. Hence,

$$E[(X+Y)^4|X-Y] = E((X+Y)^4) = E(X^4+4X^3Y+6X^2Y^2+4XY^3+Y^4) = 3+6+3 = 12$$

Problem 9.4

Let X Y be independent $N(0; 1)$ random variables. Show that $W = X^2+Y^2$ is an exponential with parameter $\frac{1}{2}$. That is, the sum of the squares of two i.i.d. zero-mean Gaussian random variables is exponentially distributed!

Solution: We calculate the characteristic function of W. We find

$$\begin{aligned}
E[e^{sW}] &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{s(x^2+y^2)} f_{X,Y}(x,y)dxdy \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{s(x^2+y^2)} \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}} dxdy \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{(s-\frac{1}{2})(x^2+y^2)} dxdy \\
&= \frac{1}{2\pi}\int_{0}^{2\pi}\int_{0}^{+\infty} e^{(s-\frac{1}{2})r^2} rdrd\theta \\
&= \int_{0}^{+\infty} e^{(s-\frac{1}{2})r^2} rdr = \int_{0}^{+\infty} \frac{1}{2(s-\frac{1}{2})}\frac{d}{dr}\left[e^{(s-\frac{1}{2})r^2}\right] dr \\
&= \frac{1}{2(s-\frac{1}{2})}[0-1] = \frac{1}{1-2s}
\end{aligned}$$

which is the moment generating function of an exponential random variable with parameter $\frac{1}{2}$.