UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probablity and Random Processes

**Instructions and review for Midterm #2**
Spring 2006

# Exam format

- The second midterm will be held in class on Thursday, April 13. You are permitted to bring a calculator, and two sides of hand-written notes on $8.5 \times 11''$ paper. (E.g., 1 sheet, both sides; or 2 sheets, 1 side only).
- Questions on the exam can be based on any material from Chapters 1 and 4, as discussed in lectures #1 through #20, and covered in homeworks #1 through #8.

# Review problems

**Problem 9.1**
For $X$, a random variable uniformly distributed between $-1$ and 1, find the density function of $Y$ where:

(a) $Y = \sqrt{|X|}$

(b) $Y = -\ln |X|$

(c) $Y = \sin(2\pi X)$

**Problem 9.2**
The receiver is an optical communications system uses a photodetector that counts the number of photons that arrive during the communication session. (The time of the communication session is 1 time unit.) The sender conveys information by either transmitting or not transmitting photons to the photodetector. The probability of transmitting is $p$. If she transmits, the number of photons $X$ that she transmits during the session has a Poisson PMF with mean $\lambda$ per time unit. If she does not transmit, she generates no photons.

Unfortunately, regardless of whether or not she transmits, there may still be photons arriving at the photodetector because of a phenomenon called shot noise. The number $N$ of photons that arrive because of the shot noise has a Poisson PMF with mean $\mu$. $N$ and $X$ are independent. The total number of photons counted by the photodetector is equal to the sum of the transmitted photons and the photons generated by the shot noise effect.

(a) What is the probability that the sender transmitted if the photodetector counted $k$ photons?

(b) Before you know anything else about a particular communication session, what is your least squares estimate of the number of photons transmitted?

(c) What is the least squares estimate of the number of photons transmitted by the sender if the photodetector counted $k$ photons?

(d) What is the best *linear* predictor for the number of photons transmitted by the sender as a function of $k$, the number of the detected photons?

**Problem 9.3**

Computers have subroutines that can generate experimental values of a random variable $X$ that is uniformly distributed in the interval $[0, 1]$. Such a subroutine can be used to generate experimental values of a continuous random variable with given CDF $F(y)$ as follows: Each time $X$ takes a value $x \in (0, 1)$, we generate the unique value $y$ for which $F(y) = x$. (We can neglect the zero probability event that $X$ takes the value 0 or 1.)

(a) Show that the CDF $F_Y(y)$ of the random variable $Y$ thus generated is indeed equal to the given $F(y)$.

(b) Describe how the procedure can be used to simulate an exponential random variable with parameter $\lambda$.

**Problem 9.4**

Your friend Bob took EE 126 last fall and now supplements his income by gambling. When he visits the casino, he plays blackjack for $Y$ hours, where $Y$ is an exponentially-distributed random variable with mean 1. Dealers are changed every hour after he starts play and, for a given dealer, the rate of his earnings (in thousands of dollars per hour) is described by a Gaussian random variable with mean $\mu$ and variance $\sigma^2$.

Let us denote the integer part of a real number $y$ by $\lfloor y \rfloor$ so, e.g., $\lfloor 2.8 \rfloor = 2$. Then, Bob's total earnings $X$ is given by

$$X = \sum_{k=1}^{S} Z_k + Z_{S+1}T,$$

where $\{Z_k\}$ is an i.i.d. sequence of Gaussian random variables with mean $\mu$ and variance $\sigma^2$ (independent of $Y$), $S = \lfloor Y \rfloor$, and $T = Y - \lfloor Y \rfloor$.

You would like to know whether you should follow Bob's footsteps, so you conduct a probabilistic analysis of his earnings.

(a) Find $\mathbb{E}[X]$.

(b) Find $p_S(k)$, the PMF of $S$, for $k = 0, 1, \ldots$.

(c) Find $f_{T|S}(t|k)$, the conditional PDF of $T$ given $S$, for $k = 0, 1, \ldots$.

You realize that, although it is difficult to obtain a description of the exact distribution of $X$, you can bound $X$ by

$$U = \sum_{k=1}^{S+1} Z_k$$

and

$$V = \sum_{k=1}^{S} Z_k$$

from above and below, respectively.

(d) Find the PDFs or transforms of $U$ and $V$.

**Problem 9.5**

An absent-minded professor schedules two student appointments for the same time; unfortunately, the professor is only able to meet with one student at a time. The appointment durations are independent and exponentially distributed with mean thirty minutes. Suppose that the first student arrives on time, but the second student arrives 5 minutes late. Determine the expected value and the variance of the time between the arrival of the first student and the departure of the second student (**Note:** The expected value is NOT 60 minutes nor 65 minutes.)

**Problem 9.6**

Every night at closing time, Virginie the chef leaves her restaurant and walks in a straight line down the street such that, at time $t$ after she starts walking, her position $X_t$ has a Gaussian distribution with mean $vt$ and variance $\sigma_X^2 t$. On most nights, Virginie is accompanied by her cat, whose position at time $t$, $Z_t$, is such that $Z_t - X_t = Y_t$, where $Y_t$ is a Gaussian-distributed random variable with mean 0 and variance $\sigma_Y^2$. The random variables $X_t$ and $Y_t$ are independent.

(a) Find the PDF of the cat's position $Z_t$ as a function of $t$. What is the correlation coefficient between the cat's position and Virginie's position as a function of $t$?

(b) On one night, at time $\tau$ after closing time, Virginie's cat is observed at position $z$. What is the linear least-squares estimate of Virginie's position at time $\tau$ given that $Z_\tau = z$?

(c) Show that the Bayes least-squares estimate of Virginie's position at time $\tau$ given $Z_\tau = z$ is the same as the linear least-squares estimate you obtained in part (b).