EECS 126: Probability and Random Processes

Problem Set 9

Due: Thurs, April 22, 2004

1. Consider again the wireless communication system in the last question of HW. 8. Instead of sending one bit of information, we would like to send a continuous RV X. (Think of this as an analog signal like voice or video.) We model $X \sim N(0, a^2)$.

a) Find the minimum mean square estimator of X given the observations at the two antennas.

b) What is the associated mean-square error? How much power do you save by using the extra antenna?

c) Redo part (a) if the noise variances are different at the antennas, say σ_1^2 and σ_2^2. Does your rule make intuitive sense?

2. a) Verify the formula given in the lecture for estimating X from Y where $Y = X + Z$ and $X \sim N(\mu, a^2)$ and $Z \sim N(0, \sigma^2)$ are independent.

b) Suppose we have an unknown RV $X \sim N(0, 1)$ we want to estimate and we are given a sequence of noisy observations $Y_i = X + Z_i, i = 1, 2, \ldots, n$, where $Z_i \sim N(0, \sigma^2$ and are i.i.d. We would like to develop an estimator for X that we can recursively update as we collect more and more data. Here is one possibility. For concreteness let us consider the case $n = 2$. First, we observe Y_1 and perform MMSE estimation of X from Y_1. Let $\hat{X}_1$ be the resulting estimate and ϵ_1 be the associated error. Now we observe Y_2. We estimate X from Y_2 by "pretending" that X is Gaussian with mean $\hat{X}_1$ and variance ϵ_1. This gives a second estimate $\hat{X}_2$.

i) Express $\hat{X}_2$ in terms of $\hat{X}_1$ and other quantities.

ii) What is the actual mean-square error of $\hat{X}_2$?

iii) Is this equal to, smaller than or larger than the MMSE of estimating X from Y_1 and Y_2 *simultaneously*?

iv) Bonus: can you explain your answer in part (iii)?

3. (a) State Markov's inequality.

(b) State Chebyshev's inequality and prove it from the Markov's inequality.

(c) Let X be a rv and $M_X(s) = E[e^{sX}]$ be its moment generating function. The Chernoff bound states that for any a,

$$P(X > a) \leq \frac{M_X(s)}{e^{sa}}$$

and this holds for any $s > 0$. By applying Markov's inequality to a suitable function of X, prove this bound.

(d) Apply all three inequalities to bound $P(X > 10)$ for X a unit mean exponential rv. Which bound is the tightest? (For the Chernoff bound, you may pick an s that you like or you can try to find the tightest bound by optimizing over all s.)

4. Let X be a rv which takes on three values $1, 2, 3$. Plot on MATLAB both the pmf and cdf of $1/\sqrt{n}\sum_{i=1}^{n} X_i$ for $n = 5, 10, 20$ and give evidence why it looks more and more Gaussian.

5. Consider the queueing example we considered in class but suppose the packet arrival process is no longer Bernouilli but is more bursty, i.e. when packets arrive they tend to arrive in a burst.

(a) Why do you think the Bernouilli arrival process does not capture the burstiness of the traffic?

(b) Give a two-state Markov chain to model the bursty arrival process. What range of values do the transition probabilities have to satisfy to capture the burstiness?

(c) Represent the entire queueing system as a Markov chain. Define carefully the state space and the transition probabilities.