Problem 1. The Viterbi algorithm.
The telephone line can be understood as an simple inter-symbol interference (ISI) channel. Specifically, the modem send independent discrete symbols $\{X_n, n = 1, 2, \ldots\}$ that takes value 1 and 0 with equal probability; on the receiver side what the modem observed is $\{Y_n, n = 1, 2, \ldots\}$

$$Y_n = X_n + X_{n-1} + W_n,$$

where $W_n, n = 1, 2 \ldots$ are i.i.d. standard Gaussian noises. As seen, there is interference among the receiving signals $\{Y_n\}$. It is known that $\{Y_n, n = 1, 2, \ldots\}$ can be modeled as a Hidden Markov chain.
a) What is the Markov chain that $\{Y_n\}$ relies on? What is the "state" there?
b) Suppose now in a specific scheme, we decode when we receive $n = 4$ receiving symbols, e.g. we try to decode X_1, X_2, X_3, X_4 when we received $Y_1 = 0.4, Y_2 = 1.2, Y_3 = 0.3, Y_4 = 0.5$. What is the optimal detection rule? Design a Viterbi algorithm to implement the optimal detection rule, and work out your detected values for X_1, X_2, X_3, X_4. Compare the detected values you get for X_1 based only on Y_1 and all Y_i, are they the same?
c) Suppose now you receive another $Y_5 = 0.4$, do you need to restart all the decoding process? Give out the updated detected sequence $X_i, i = 1 \sim 5$.
d) How many computations and comparisons you need for decoding by Viterbi algorithm. How are they scale with respect to n?

Problem 2.
a) In class we showed for a two state Markov chain with transition probability $p = 0.5, q = 0.5$ before time 100, and changing $p = 0.1, q = 0.3$ after time 100, the whole process is not stationary. Plot a sample path to get a visual idea on why it is true and how to explain it intuitively.
b) Here is a process $\{X_n, n = 1, 2, \ldots\}$, with

$$X_n = \begin{cases} 0, & \text{if } n \text{ is even;} \\ \text{takes value } \pm 1 \text{ with equal probability,} & \text{otherwise.} \end{cases}$$

Is $\{X_n\}$ stationary?
c) For the $\{X_n\}$ in part b), we define another process $\{Y_n\}$ as

$$Y_n = X_{n+t},$$

where t is uniformly distributed in $\{0, 1\}$. Then is $\{Y_n\}$ stationary?