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UNIVERSITY OF CALIFORNIA

College of Engineering

Department of Electrical Engineering and Computer Sciences

Professor Evans Spring 1999

EECS 126 — FINAL EXAM

19 May 1999, Wednesday, 8:00 - 11:00 a.m.

[20 pts.] 1. You are told that behind a curtain there are two coins. The probability of tossing a head for Coin A is 0.75 and for Coin B is 0.25. A person behind the curtain randomly follows one of two coin tossing strategies:

- In Strategy One, one of the coins is selected at random and the selected coin is tossed 10 times.
- In Strategy Two, one of the coins is selected at random and tossed once. The process is then repeated 10 times.

When the experiment is finished, it is revealed that exactly 7 heads were tossed in the 10 throws. What is the probability that Strategy One was followed, given this information?

[20 pts.] 2. In the major project of one of your favorite classes, you design and build a modem for a specified communications channel. The project is graded by sending 1000 bits through your modem and noting the number of errors. Each error decreases your mark out of one hundred by twenty points until five or more errors occur, in which case you receive a big fat zero. After spending 4,327 hours on your project you finish with a modem that has a bit error probability of 10^{-3}.

a) Assuming that bit errors occur independently, what is your expected mark out of 100?

b) What bit error probability would be required so that you got the full 100 points with 95% probability?

Feel free to use appropriate approximations if required (like the Poisson approximation to the binomial, perhaps).

[20 pts.] 3. The number of beers consumed at a certain bar on Thursday night is a Poisson random variable with parameter λ. The parameter λ is deterministic but unknown and in order to estimate it, you go to the bar every Thursday night for t weeks and observe the number of beers consumed (making sure not to influence the observations yourself). Let these observed random variables be $N_1, \ldots, N_t$ and assume that they are independent.

The maximum likelihood (ML) estimate of λ given $N_1 = n_1, \ldots, N_t = n_t$, $\hat{\lambda}(n_1, \ldots, n_t)$, is the value of λ that maximizes the likelihood function

$$p_{N_1, \ldots, N_t}(n_1, \ldots, n_t; \lambda)$$

which is the joint probability mass function of $N_1, \ldots, N_t$ parameterized by λ. The ML estimator is the random variable $\hat{\lambda}(N_1, \ldots, N_t)$.

a) Show that the ML estimator of λ is

$$\hat{\lambda} = \frac{1}{t} \sum_{i=1}^{t} N_i$$

b) Using the moment generating functions, prove that $\sum_{i=1}^{t} N_i$ is a Poisson random variable with mean $t\lambda$.

c) Write down an exact expression for the probability that $\hat{\lambda}$ is within 10% of the actual value of λ

$$P(|\hat{\lambda} - \lambda| < 0.1\lambda)$$

d) Use the Central Limit Theorem to write down an approximate expression for this probability in terms of the Q function.

[20 pts.] 4. A sample of a speech signal, X, is modelled as a normal random variable with zero mean and unit variance. Rather than observing X, you observe a coarsely quantized version

$$Y = \mathrm{sign}(X) = \begin{cases} -1, & \text{if } X < 0 \\ +1, & \text{if } X \geq 0 \end{cases}$$

a) Write down the probability mass function of Y.

b) Write down the conditional probability mass function of Y given $X = x$.

c) Calculate $E[Y]$, $var(Y)$, and $cov(X, Y)$.

d) Determine the LMMSE estimator of X given Y and the corresponding MSE.

e) Write down the conditional probability density function of X given $Y = 1$.

f) Determine the MMSE estimator of X given Y.

g) Determine the MAP estimator of X based on Y.

Note that

$$\int_0^{\infty} g(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} g(x)dx$$

whenever g is an even function.

[20 pts.] 5. The distance to a submarine is modelled as a Gaussian random variable D, with mean μ_D and standard deviation σ_D. You wish to estimate the distance to the submarine based on measurements obtained by an active sonar. The measurements are modelled as

$$Y_i = D + W_i, \qquad i = 1, \ldots, n$$

where $W_1, \ldots, W_n$ are independent and identically distributed Gaussian random variables with zero mean and standard deviation σ_W that are independent of D.

There is a cost associated with each measurement because it increases the likelihood of the submarine discovering that it is being tracked. Assume that each measurement made costs R units.

At the same time you wish to obtain as accurate an estimate as possible and clearly, the more measurements, the better.

Your mission (should you choose to accept it) is to find a value for n and an estimator for D based on $Y_1, \ldots, Y_n$ so as to minimize the cost function

$$C(n) = nR + \Delta(n)$$

where $\Delta(n)$ is the mean squared error when n observations are made.

You should first work out the MMSE estimator for D based on $Y_1, \ldots, Y_n$ and the MMSE, $\Delta(n)$. You can then find the value of n which maximizes $C(n)$. If you like, you can perform the optimization treating n as a positive real number, although an alternative technique is to find the value of n where $C(n+1)/C(n)$ changes from being less than one to being greater than one.

[20 pts.] 6. In a single-user direct-sequence spread-spectrum communication system the received signal is modelled as

$$Y(i) = s_1(i)X_1 + W(i), \qquad i = 1, \ldots, n$$

where X_1 is the data symbol that takes -1 and 1 with equal probability, where $W(1), \ldots, W(n)$ are independent and identically distributed Gaussian random variables with zero mean and variance σ^2, and $s_1(1), \ldots, s_1(n)$ is the known signature sequence. Assume $X_1, W(1), \ldots, W(n)$ are independent. Let $R_1 = \sum_{i=1}^{n} s_1(i)Y(i)$.

a) Show that the MAP detector for X_1 based on $Y(1), \ldots, Y(n)$ reduces to the form

$$X^* = \mathrm{sign}(R_1) = \begin{cases} -1, & \text{if } R_1 < 0 \\ +1, & \text{if } R_1 > 0 \end{cases}$$

b) Determine the minimum probability of error.

Bonus Question (20 points)

Following on from Question 6, in a two-user direct-sequence spread-spectrum communication system the received signal is modelled as

$$Y(i) = s_1(i)X_1 + s_2(i)X_2 + W(i), \qquad i = 1, \ldots, n$$

where X_1 and X_2 are the data symbols for users 1 and 2 that take values -1 and 1 with equal probability, and $s_1(1), \ldots, s_1(n)$ and $s_2(1), \ldots, s_2(n)$ are the known signature sequences of users 1 and 2. The background noise $W(1), \ldots, W(n)$ is as in Question 6 and $X_1, X_2, W(1), \ldots, W(n)$ are independent. Let $R_1 = \sum_{i=1}^{n} s_1(i)Y(i)$, $R_2 = \sum_{i=1}^{n} s_2(i)Y(i)$, and $\rho = \sum_{i=1}^{n} s_1(i)s_2(i)$.

a) Show that the MAP detector for jointly detecting X_1 and X_2 based on $Y(1), \ldots, Y(n)$ reduces to

$$(X_1^*, X_2^*) = \arg \max_{(X_1 X_2)} R_1X_1 + R_2X_2 - \rho X_1X_2$$

b) Derive the decision rule that minimizes the error probability for user 1 alone.

c) Show that when $\rho = 0$, so that the users are orthogonal, both of the above detectors reduce to single-user receivers.