1. Uncorrelated & Independent

(a) If $X$ and $Y$ are uncorrelated, $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$.

(b) If $X_1, \ldots, X_n$ are uncorrelated, $\text{var}(X_1 + \cdots + X_n) = \sum_{i=1}^n \text{var}(X_i)$.

(c) Show that independent random variables are uncorrelated.

(d) Find an example, where a pair of random variables are uncorrelated but not independent.

Solution:

(a) As $X$ and $Y$ are uncorrelated, $\text{cov}(X, Y) = 0$, hence, $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2 \text{cov}(X, Y) = \text{var}(X) + \text{var}(Y)$.

(b) Observe the fact that if $X_1, \ldots, X_n$ are uncorrelated, $X_1 + \cdots + X_{n-1}$ is uncorrelated with $X_n$. Hence, $\text{var}(X_1 + \cdots + X_n) = \text{var}(X_1 + \cdots + X_{n-1}) + \text{var}(X_n)$. Iteratively, further applying this on $X_1, \ldots, X_{n-1}$ and so on, we get the result.

(c) Since $X$ and $Y$ are independent, $\mathbb{E}(f(X)g(Y)) = \mathbb{E}(f(X))\mathbb{E}(g(Y))$, for any functions $f, g$. Now pick $f(X) = X - \mathbb{E}(X)$ and $g(Y) = Y - \mathbb{E}(Y)$. We have

$$\text{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = 0 \cdot 0 = 0.$$  
Hence they are uncorrelated.

(d) Consider $X \sim \mathcal{N}(0, 1)$ and $Y = ZX$, where $Z \in \{1, -1\}$ with probability $1/2, 1/2$ ($Z$ is called a Rademacher random variable). Now since $\mathbb{E}(X) = 0$, $\mathbb{E}(X)\mathbb{E}(Y) = 0$. Also, $\mathbb{E}(XY) = \mathbb{E}(ZX^2) = 0$ (from the distribution of $Z$). So, $X$ and $Y$ are uncorrelated. But observe that $Y$ is generated based on $X$, so $Y$ cannot be independent of $X$.

2. Second Moment Method

Consider a non-negative RV $Y$, with $\mathbb{E}(Y^2) < \infty$. Show that

$$\mathbb{P}(Y > 0) \geq \frac{\mathbb{E}(Y)^2}{\mathbb{E}(Y^2)}.$$  

Hint: Use Cauchy-Schwarz on $Y 1_{\{Y>0\}}$.

Solution:
Applying Cauchy-Schwarz on \( Y_1 \{ Y > 0 \} \),

\[
\mathbb{E}(Y_1 \{ Y > 0 \})^2 \leq \mathbb{E}(Y^2)\mathbb{E}(1^2 \{ Y > 0 \}) = \mathbb{E}(Y^2)\mathbb{P}(Y > 0)
\]

where we use the fact that, since the indicator function is a \( \{0, 1\} \)-valued function, squaring will not make a difference. Also, we claim that, for non-negative \( Y \), \( Y_1 \{ Y > 0 \} \) equals \( Y \). For \( Y > 0 \), \( Y_1 \{ Y > 0 \} = Y \), and for \( Y = 0 \), \( Y_1 \{ Y > 0 \} = 0 = Y \). Hence, the claim follows.

3. **Conditioning on the Minimum of Uniforms**

If \( X \) and \( Y \) are independent Uniform\([0, 1]\), show that

\[
\mathbb{E}(Y \mid \min\{X, Y\}) = \frac{1}{4} + \frac{3}{4} \min\{X, Y\}.
\]

**Solution:**

We consider two cases: (i) \( Y = \min\{X, Y\} \), i.e., \( Y < X \), and (ii) \( X = \min\{X, Y\} \), i.e., \( X < Y \). Since \( X \) and \( Y \) have the same distribution, from symmetry, the occurrences of case (i) and (ii) are equiprobable with probability 1/2. We compute \( \mathbb{E}(Y \mid \min\{X, Y\}) \) under these 2 cases.

Case (i): \( \mathbb{E}(Y \mid \min\{X, Y\} = Y) = \mathbb{E}(Y \mid Y) = Y = \min\{X, Y\} \).

Case (ii): Since \( X < Y \), \( Y \sim \text{Uniform}[X, 1] \), hence

\[
\mathbb{E}(Y \mid \min\{X, Y\} = X) = \frac{X + 1}{2} = \frac{1 + \min\{X, Y\}}{2}.
\]

Combining everything,

\[
\mathbb{E}(Y \mid \min\{X, Y\}) = \frac{1}{2} \min\{X, Y\} + \frac{1 + \min\{X, Y\}}{4}.
\]