UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probablity and Random Processes

Problem Set 11
Fall 2007

Issued: Friday, November 30, 2007 **Due:** NOT TO BE HANDED IN FOR GRADING

Reading: Bertsekas & Tsitsiklis, Chapter 5, §6.3, 6.4, 7.1–7.4

Problem 11.1
Two thimbles (like tiny cups) are under a dripping roof. At the end of each second, thimble A receives 1 drop of water with probability 1, and thimble B receives 1 drop with probability 2/3 and 0 drops otherwise. By a complicated automatic mechanism, right before a 4th drop lands in thimble A, *both* thimbles are emptied. While the thimbles are being emptied, they miss catching the drops that would have otherwise landed inside the thimbles.

(a) Set up a Markov chain model, and draw the state transition diagram, and indicate the transition probabilities.

(b) If both thimbles were empty when you started watching, what is the probability that both thimbles contain exactly 1 drop after exactly 10,001 seconds?

Problem 11.2
At the UC Berkeley book store, there is only one cashier. Due to the limited space, she allows only M customers to line up before her at any time. If a customer finds there are M customers there including the one being served by the cashier, he will leave the book store immediately. Every minute, exactly one of the following occurs: (i) one new customer arrives with probability p; or (ii) one existing customer leaves with probality kq, where k is the number of customers in the book store; or (iii) no new customer arrives and no existing customer leaves with probability $1-p-kq$ if there is at least one customer in the book store and with probability $1-p$ otherwise.

(a) This problem can be modeled as a birth-death process (see end of §6.3). Define appropriate states and draw the transition probability graph.

(b) After the book store has been open for a long time, you walk into the book store. Calculate how many customers you expect to see in line.

Problem 11.3
Sam and Pat are playing foosball. When they begin, the score is 0-0. To make things interesting, if the score ever becomes tied, it is instantly reset to 0-0. Starting from any score, the probability that Sam gets the next point is $\frac{1}{3}$.

(a) Suppose the game stops when one player's score reaches 2. Draw an appropriate Markov chain that describes the game, and identify all transient, recurrent, and periodic states. Find $\mathbb{P}$[Pat wins].

(b) Now suppose instead that the game stops when a total of 3 points have been scored (note that this stopping condition does not explicitly depend on the score). The score still resets to 0-0 when the game is tied, and the player with the most points when the game ends wins. Draw an appropriate Markov chain that describes the game.

Problem 11.4

Let X be the height in meters of a randomly selected Canadian. Bo is interested in estimating $h = \mathbb{E}[X]$. Being sure that no Canadian is taller than 3 meters, Bo decides to use 1.5 meters as a conservative (large) value for the standard deviation of X. To estimate h, Bo compute the average H of the heights of n Canadians that he selects at random.

(a) Compute $\mathbb{E}[H]$ and $\text{var}(H)$ in terms of h and Bo's 1.5 meter bound for $\text{std}(X)$.

(b) Compute the minimum value of n (with $n > 0$) such that the standard deviation of H will be less than 0.01 meters.

(c) Say Bo would like to be 99% sure that his estimate is within 5 centimeters of the true average height of Canadians. Using the Chebyshev inequality, calculate the minimum value of n required.

(d) If we agree that no Canadians are taller than three meters, why is it correct to use 1.5 meters as an upper bound on the standard deviation for X, the height of any Canadian selected at random?

Problem 11.5

Consider a normal variable $Z \sim N(\mu, \sigma^2)$.

(a) Use the Chernoff approach to show that $\ln \mathbb{P}[Z \geq \mu + \epsilon] \leq \min_{t>0} \left[\frac{\sigma^2 t^2}{2} - \epsilon t\right]$ for any $\epsilon > 0$. Conclude that $\mathbb{P}[Z \geq \mu + \epsilon] \leq \exp(-\frac{\epsilon^2}{2\sigma^2})$.

(b) Let $Z_1, Z_2, \ldots$ be independent and identically distributed as $Z(0, \sigma^2)$. Show that the probability $\mathbb{P}[\max\{Z_1, \ldots, Z_n\} \geq \sqrt{3\sigma^2 \log n}] \to 0$, as $n \to +\infty$. (*Hint:* Union bound combined with part (a) could be useful.)

Problem 11.6

We are laying out 25 plastic planks lengthwise, trying to make a path of about 1000 meters. The plastic planks are made in molds, and any variation in the lengths of the planks is due entirely to variation between different molds. The length in meters, X, of any particular mold used for making planks is independent of the length of all other molds. X is uniformly distributed between $40 - \sqrt{3}$ and $40 + \sqrt{3}$ meters. X has an expected value of 40 meters and a standard deviation of 1 meter. Assume every plank has exactly the same length as its mold. What is the probability that the resulting path will be within 1000 ± 7.5 meters if we use 25 planks:

(a) all made from the same mold?

(b) each made from a different mold?

Explain the difference between the answers to parts (a) and (b).