UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probablity and Random Processes

Problem Set 9
Fall 2006

Issued: Thursday, November 2, 2006 **Due:** Monday, November 13, 2006

Reading: For this problem set: §4.4, 4.6, 4.7

Problem 9.1

Continuous random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{x}{8}, & \text{for } 1 \le x \le 3 \text{ and } 0 \le y \le 2. \\ 0, & \text{otherwise.} \end{cases}$$

Let R be defined by $R = Y/X$ and let D be the event $R \le 1/3$.

(a) Are X and Y independent? Are they independent conditioned on event D? Justify your answers.

(b) Determine the numerical value of $P(D)$.

(c) Obtain and carefully sketch the PDF's $f_X(x)$ and $f_{X|D}(x|D)$.

(d) Determine the conditional PDF for R given D has occurred.

(e) Determine an estimator $\hat{R}(y)$ of R that minimizes $\mathbb{E}[(R-\hat{R}(y))^2|Y=y]$, for all possible observed experimental values y. Provide a plot of the optimal estimator as a function of y.

Problem 9.2

Using a fair three-sided die (try to construct such a die, if you dare!), we will decide how many times to spin a fair wheel of fortune. The wheel of fortune is calibrated infinitely finely and has numbers between 0 and 1. The die has the numbers 1,2 and 3 on its faces. Whichever number results from our throw of the die, we will spin the wheel of fortune that many times and add the results to obtain random variable Y.

(a) Determine the expected value of Y.

(b) Determine the variance of Y.

Problem 9.3

Let $\underline{V} = (X, Y)$ be a pair of zero mean jointly Gaussian random variables. Let K be the *covariance matrix* of $\underline{V}$ defined as

$$K = \begin{pmatrix} \mathbf{E}[X^2] & \mathbf{E}[XY] \\ \mathbf{E}[XY] & \mathbf{E}[Y^2] \end{pmatrix}.$$

(a) Show that the joint PDF of X and Y can be written as

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{|K|}} e^{-\frac{1}{2}\underline{v}^{t}K^{-1}\underline{v}} \quad ,$$

where $\underline{v}$ is the column vector $(x,y)^{t}$, $|K|$ denotes the determinant of the matrix K. and K^{-1} its inverse.

Hint: Recall the matrix inverse formula (for $ad - bc \neq 0$)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(b) Let $Z = 2X + Y$ and $W = X - 2Y$.

(i) Are Z and W jointly Gaussian?

(ii) Find the joint PDF of Z and W.

Problem 9.4

Sambuca bottles are placed into boxes, and boxes are packed into a crate. Let X be the number of bottles in any particular box, and let N be the number of boxes in a crate. Suppose that X and N are independent, and each have the same PMF:

$$p_X(u) = p_N(u) = \begin{cases} (\frac{1}{3})(\frac{2}{3})^{u-1} & u = 1, 2, 3, ... \\ 0 & otherwise \end{cases}$$

Let T be the number of bottles in a crate.

(a) Find $\mathbb{E}[T]$.

(b) Find $\mathrm{var}(T)$.

(c) Find the transform of T, $M_T(s)$.

(d) Find the PMF of T, $p_T(t)$.

(e) Suppose we count the number of boxes in a crate, and we know that $N = n$. Find the least-squares estimate of T given $N = n$.

Problem 9.5

Consider three zero-mean random variables X, Y, and Z, with known variances and covariances. Give a formula for the linear least squares estimator of X based on Y and Z, that is, find a and b that minimize

$$\mathbb{E}\big[(X - aY - bZ)^2\big].$$

For simplicity, assume that Y and Z are uncorrelated.