UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probablity and Random Processes

Problem Set 3
Fall 2006

Issued: Thursday, September 14, 2006 **Due:** Friday, September 22, 2006

Reading: For this problem set: §1.4, 1.5, §1.6

Problem 3.1
You are lost on a university campus, where the population is entirely composed of brilliant students and absent-minded professors. The students comprise two-thirds of the population, and any one student gives a correct answer to a request for directions with probability $\frac{3}{4}$. (Assume answers to repeated questions are independent, even if the question and the person asked are the same.) If you ask a professor for directions, the answer is always false.

(a) You ask a passer-by whether the exit from campus is East or West. The answer is East. What is the probability this is correct?

For the rest of the problem (parts b through e), we are looking at conditional events:

(b) You ask the same person again, and receive the same reply. Show that the probability that this second reply is correct is $\frac{1}{2}$.

(c) You ask the same person again, and receive the same reply. What is the probability that this third reply is correct?

(d) You ask for the fourth time, and receive the answer East again. Show that the probability it is correct is $\frac{27}{70}$.

(e) Show that, had the fourth answer been West instead, the probability that East is nevertheless correct is $\frac{9}{10}$.

Your friend, Ima Nerd, happens to be in the same position as you are, only she has reason to believe a-priori that, with probability ϵ, East is the correct answer.

(f) Show that whatever answer is first received, Ima continues to believe that East is correct with probability ϵ.

(g) Show that if the first two replies are the same (that is, either WW or EE), Ima continues to believe that East is correct with probability ϵ.

(h) Show that after three like answers, Ima will calculate as follows (in the obvious notation):

$$\mathbb{P}(\text{East correct}|EEE) = \frac{9\epsilon}{11 - 2\epsilon}, \qquad \mathbb{P}(\text{East correct}|WWW) = \frac{11\epsilon}{9 + 2\epsilon}.$$

Problem 3.2
Imagine a drunk tightrope walker, who manages to keep his balance, but takes a step forward with probability p and takes a step back with probability $(1 - p)$.

(a) What is the probability that after 2 steps the tightrope walker will be at the same place on the rope?

(b) What is the probability that after three steps, the tightrope walker will be one step forward from where he began?

(c) Given that after three steps he has managed to move ahead one step, what is the probability that the first step he took was a step forward?

Problem 3.3

How many 6-word sentences can be made using each of the 26 letters of the alphabet exactly once? A word is defined as a nonempty (possibly jibberish) sequence of letters.

Problem 3.4

Suppose there are n couples and it is desirable that men and women alternate when seated at a circular table. Without loss of generality, label the seats $1, 2, \ldots, 2n$ clockwise and dictate that seat 1 is occupied by a woman; note that this determines the sex of the occupant of every other seat. For $1 \leq k \leq 2n$, let A_k be the event that seats k and $k+1$ are occupied by one of the couples, identifying seat $2n+1$ with seat 1. We wish to ultimately prove that the probability that nobody is sitting next to his or her partner is

$$\mathbb{P}\left(\bigcap_{k=1}^{2n} A_k^c\right) = \frac{2}{(n-1)!}\sum_{k=0}^{n}(-1)^k \binom{2n-k}{k}\frac{(n-k)!}{2n-k} \quad .$$

The following steps guide you through the proof.

(a) Show that $\mathbb{P}(A_k) = n\left(\frac{(n-1)!}{n!}\right)^2$.

(b) Show that, if $1 \leq i < j \leq 2n$, then

$$\mathbb{P}(A_i \cap A_j) = \begin{cases} n(n-1)\left(\dfrac{(n-2)!}{n!}\right)^2 & , \quad |i-j| \neq 1 \text{ and } (i,j) \neq (1,2n) \\ 0 & , \quad \text{otherwise} \end{cases} \quad .$$

(c) Generalizing part (b), determine an expression for the probability $\mathbb{P}\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right)$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq 2n$.

(d) Letting $S_{k,n}$ denote the number of ways of choosing k non-overlapping pairs of adjacent seats out of $2n$ seats, utilize results from parts (a)-(c) and show that

$$\mathbb{P}\left(\bigcap_{k=1}^{2n} A_k^c\right) = \sum_{k=0}^{n}(-1)^k \left(\frac{(n-k)!}{n!}\right) S_{k,n} \quad .$$

Hint: use the inclusion-exclusion principle (see p. 54).

(e) Show that, as defined in part (d),

$$S_{k,n} = \binom{2n-k}{k}\frac{2n}{2n-k} \quad .$$