Problem 1 *True or False. Prove or show a counterexample:*

a. *True. The pdf of X condition on $Y = 0$ is $\frac{f_{XY}(x,0)}{f_Y(0)} = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. The level sets of $f_{XY}(x,y)$ are circles centered at $(0,0)$, so $\forall (x,y) \in \mathcal{R}^2$:*

$$\begin{aligned} f_{XY}(x,y) &= f_{XY}(\sqrt{x^2+y^2}, 0) \\ &= f_Y(0)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2+y^2}{2}} \end{aligned}$$

The integral of $f_{XY}(x,y)$ over $\mathcal{R}^2$ is 1, meanwhile the integral of $\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$ over $\mathcal{R}^2$ is 1, so $f_Y(0) = \frac{1}{\sqrt{2\pi}}$. So

$$f_{XY}(x,y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$$

X, Y are jointly Gaussian.

b. *True. Sketch of the proof:*

$\forall A_1, A_2 \subseteq \mathcal{R}$

$$\begin{aligned} P(Z_1 \in A_1, Z_2 \in A_2) &= P(f(X) \in A_1, f(Y) \in A_2) \\ &= P(X \in f^{-1}(A_1), Y \in f^{-1}(A_2)) \\ &= P(X \in f^{-1}(A_1))P(Y \in f^{-1}(A_2)) \\ &= P(Z_1 \in A_1)P(Z_2 \in A_2) \end{aligned}$$

c. *False. A counterexample: let X be a continuous random variable, let $f(x) = -x$, then:*

$$P(f(X) \leq f(\beta)) = P(-X \leq -\beta) = P(X \geq \beta) = 1 - P(X \leq \beta) = 0.9$$

Problem 2 *Random Processes*

a. *Let X_i be the number of people on the i'th bus. Let T_i be the time the i'th bus arrives, then $Y_i = T_i - T_{i-1}$ is a exponential random variable with parameter $\lambda = \frac{1}{2}$, let $f_Y(t) = \frac{1}{2}e^{-\frac{t}{2}}, t \geq 0$. In an interval of length t, the number of arrivals of the taxis is a Poisson random variable with parameter t . Noticing that the two processes are independent,*

then

$$\begin{aligned}
P(X_i = n) &= \int_{t=0}^{\infty} P(X_i = n | Y_i = t) f_Y(t) dt \\
&= \int_{t=0}^{\infty} e^{-t} \frac{t^n}{n!} (\frac{1}{2} e^{-\frac{t}{2}}) dt \\
&= \frac{1}{2(n!)} \int_{t=0}^{\infty} t^n e^{-\frac{3t}{2}} dt \\
&= {}_{(a)} \frac{1}{2(n!)} \frac{n!}{(\frac{3}{2})^{n+1}} \\
&= \frac{1}{2} (\frac{2}{3})^{n+1}
\end{aligned}$$

(a): integral by parts and so on.

b.1 *The process is not i.i.d, but it is Markov. The transition matrix is :*

$$P = \begin{pmatrix} \frac{1+p}{2} & \frac{1-p}{2} \\ \frac{1-p}{2} & \frac{1+p}{2} \end{pmatrix}$$

b.2 *X_n's are Markov, so $Z_n = 0$ if $X_n = X_{n-1}$, the probability is $\frac{1+p}{2}$, $Z_n = 1$ if $X_n \neq X_{n-1}$, the probability is $\frac{1-p}{2}$. Z_n's are i.i.d Bernoulli random variables. So Z_n is a Bernoulli random process, the parameter is $\frac{1-p}{2}$*

Problem 3 *Markov Chains*

a. *Transient: C*
Recurrent: A, B, D, E, F, G
Aperiodic: A, B, D
Periodic: E, F, G all with period 3.

b. *For aperiodic class $[D]$, the steady-state distribution is $\pi_{[D]} = (1)$*
For aperiodic class $[A, B]$, the steady-state distribution is $\pi_{[A,B]} = (0.5, 0.5)$

c.

c.1 *The steady-state distribution is $\pi_{[A,B]} = (\frac{3}{7}, \frac{4}{7})$*
It is unique because A and B form an aperiodic recurrent class, and according to the steady-state convergence theorem, the distribution is unique.

c.2

$$
\begin{aligned}
&P(Y=1)=0.4\\
&P(Y=2)=0.6\times 0.4\\
&P(Y=1)=0.6^2\times 0.4\\
&.....\\
&P(Y=n)=0.6^{n-1}\times 0.4
\end{aligned}
$$

So Y is a geometric random variable with parameter $p=0.4$

$$E(Y)=\frac{1}{p}=2.5 \qquad Var(Y)=\frac{1-p}{p^2}=3.75$$

c.3 First, we write the the transition matrix P as:

$$P=\begin{pmatrix}0.6 & 0.4\\ 0.3 & 0.7\end{pmatrix}=V\Lambda V^{-1}$$

Where Λ is the eigenvalue matrix of P, and V is a right eigenvector matrix of P.

$$\Lambda=\begin{pmatrix}1 & 0\\ 0 & 0.3\end{pmatrix} \qquad V=\begin{pmatrix}1 & 4\\ 1 & -3\end{pmatrix} \qquad V^{-1}=\begin{pmatrix}\frac{3}{7} & \frac{4}{7}\\ \frac{1}{7} & -\frac{1}{7}\end{pmatrix}$$

The n'th order transition matrix P^n is :

$$P^n=(V\Lambda V^{-1})^n=V\Lambda^n V^{-1}=V\begin{pmatrix}1 & 0\\ 0 & 0.3^n\end{pmatrix} \qquad V^{-1}$$

So $P(X_n=B|X_0=A)=P^n(1,2)=\frac{4}{7}(1-0.3^n)$

c.4 Let X_T be the total money we pay at time T. Let T_A be the number of total times we stay at A at time T, T_B be the number of total times we stay at B at time T. Then $T_A+T_B=T$ and $X_T=T_A+2T_B$.

$$
\begin{aligned}
E(X_T) &= E(T_A+2T_B)\\
&= T+E(T_B)\\
&= T+\sum_{n=1}^{T}P(X_n=B|X_1=A)\\
&= T+\sum_{n=1}^{T}\frac{4}{7}(1-0.3^{n-1})\\
&= \frac{11T}{7}+\frac{40}{49}(1-0.3^T)
\end{aligned}
$$

Problem 4 *Rolling Fair Dice*

a. *You can bound the probability with the Chernoff bound, Markov inequality or the Chebyshev inequality. I will bound it using Chevyshev inequality. Write* $Y_n = \frac{\sum_{i=1}^n X_i}{n}$. *Then* $E(Y_n) = E(X_1) = 3.5$, $Var(Y_n) = \frac{Var(X_1)}{n} = \frac{35}{12n}$. *So the Chebysehve inequality gives:*

$$\begin{aligned} P(Y_n \geq 5) &\leq P(|Y_n - 3.5| \geq 1.5) \\ &\leq \frac{Var(Y_n)}{1.5^2} \\ &= \frac{35}{27n} \end{aligned}$$

b. *When* n *is large, we can use the Central Limit Theorem: the CDF of* $\frac{\sum_{i=1}^n X_i - 3.5}{\sqrt{\frac{35}{12}n}}$ *converges to the standard normal CDF. Let* N *be a standard normal random variable. So approximately :*

$$\begin{aligned} P(\frac{\sum_{i=1}^n X_i}{n} \in [3.5, 3.6]) &= P(\frac{\sum_{i=1}^n X_i - 3.5}{\sqrt{\frac{35}{12}n}} \in [0, 0.1 \times \sqrt{\frac{12}{35}n}]) \\ &= P(N \in [0, \sqrt{\frac{12n}{3500}}]) \\ &= Q(\sqrt{\frac{12n}{3500}}) - Q(0) \\ &= Q(\sqrt{\frac{12n}{3500}}) - \frac{1}{2} \end{aligned}$$

c. *Write the outcome sequence as* $X_1X_2....X_{100}$, 50 *of the* X_i*'s are* T, *the rest are* H. *There are* $\binom{100}{50}$ *such sequences. Among them there are* $\binom{96}{50}$ *sequences with* $X_1X_2X_3X_4 = TTTT$. *So the probability that the first 4 flips are all heads is*

$$\frac{\binom{96}{50}}{\binom{100}{50}} = \frac{\frac{96!}{50!46!}}{\frac{100!}{50!50!}} = \frac{50 \times 49 \times 48 \times 47}{100 \times 99 \times 98 \times 97} = 0.0587$$