Problem 1  True or False. Prove or show a counterexample:

a. True. The pdf of $X$ condition on $Y = 0$ is $f_{X|Y}(x,0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. The level sets of $f_{XY}(x,y)$ are circles centered at $(0,0)$, so $\forall (x,y) \in \mathbb{R}^2$:

$$f_{XY}(x,y) = f_{XY}(\sqrt{x^2+y^2},0) = f_Y(0) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2+y^2}{2}}$$

The integral of $f_{XY}(x,y)$ over $\mathbb{R}^2$ is 1, meanwhile the integral of $\frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$ over $\mathbb{R}^2$ is 1, so $f_Y(0) = \frac{1}{\sqrt{2\pi}}$. So

$$f_{XY}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

$X,Y$ are jointly Gaussian.

b. True. Sketch of the proof:

$$\forall A_1, A_2 \subseteq \mathbb{R}$$

$$P(Z_1 \in A_1, Z_2 \in A_2) = P(f(X) \in A_1, f(Y) \in A_2) = P(X \in f^{-1}(A_1), Y \in f^{-1}(A_2)) = P(X \in f^{-1}(A_1))P(Y \in f^{-1}(A_2)) = P(Z_1 \in A_1)P(Z_2 \in A_2)$$

c. False. A counterexample: let $X$ be a continuous random variable, let $f(x) = -x$, then:

$$P(f(X) \leq f(\beta)) = P(-X \leq -\beta) = P(X \geq \beta) = 1 - P(X \leq \beta) = 0.9$$

Problem 2 Random Processes

a. Let $X_i$ be the number of people on the $i$'th bus. Let $T_i$ be the time the $i$'th bus arrives, then $Y_i = T_i - T_{i-1}$ is an exponential random variable with parameter $\lambda = \frac{1}{2}$, let $f_Y(t) = \frac{1}{2} e^{-\frac{1}{2}t}, t \geq 0$. In an interval of length $t$, the number of arrivals of the taxis is a Poisson random variable with parameter $t$. Noticing that the two processes are independent,
then

\[ P(X_i = n) = \int_{t=0}^{\infty} P(X_i = n | Y_i = t) f_Y(t) \, dt \]
\[ = \int_{t=0}^{\infty} e^{-t} \frac{t^n}{n!} \left( \frac{1}{2} e^{-\frac{t}{2}} \right) \, dt \]
\[ = \frac{1}{2(n!)} \int_{t=0}^{\infty} t^n e^{-\frac{3t}{2}} \, dt \]
\[ = \frac{1}{2(n!)} \frac{n!}{\left( \frac{3}{2} \right)^{n+1}} \]
\[ = \frac{1}{2} \left( \frac{2}{3} \right)^{n+1} \]

(a): integral by parts and so on.

b.1 The process is not i.i.d, but it is Markov. The transition matrix is:

\[ P = \begin{pmatrix} \frac{1+p}{2} & \frac{1-p}{2} \\ \frac{1-p}{2} & \frac{1+p}{2} \end{pmatrix} \]

b.2 \( X_n \)'s are Markov, so \( Z_n = 0 \) if \( X_n = X_{n-1} \), the probability is \( \frac{1+p}{2} \), \( Z_n = 1 \) if \( X_n \neq X_{n-1} \), the probability is \( \frac{1-p}{2} \). \( Z_n \)'s are i.i.d Bernoulli random variables. So \( Z_n \) is a Bernoulli random process, the parameter is \( \frac{1-p}{2} \).

Problem 3 Markov Chains

a. Transient: C
   Recurrent: A, B, D, E, F, G
   Aperiodic: A, B, D
   Periodic: E, F, G all with period 3.

b. For aperiodic class \([D]\), the steady-state distribution is \( \pi_{[D]} = (1) \)
   For aperiodic class \([A, B]\), the steady-state distribution is \( \pi_{[A,B]} = (0.5, 0.5) \)

c.

c.1 The steady-state distribution is \( \pi_{[A,B]} = \left( \frac{3}{7}, \frac{4}{7} \right) \)
   It is unique because A and B form an aperiodic recurrent class, and according to the steady-state convergence theorem, the distribution is unique.
c.2

\[ P(Y = 1) = 0.4 \]
\[ P(Y = 2) = 0.6 \times 0.4 \]
\[ P(Y = 1) = 0.6^2 \times 0.4 \]
......
\[ P(Y = n) = 0.6^{n-1} \times 0.4 \]

So \(Y\) is a geometric random variable with parameter \(p = 0.4\)

\[ E(Y) = \frac{1}{p} = 2.5 \]
\[ Var(Y) = \frac{1 - p}{p^2} = 3.75 \]

c.3 First, we write the transition matrix \(P\) as:

\[ P = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} = VA^{-1} \]

Where \(\Lambda\) is the eigenvalue matrix of \(P\), and \(V\) is a right eigenvector matrix of \(P\).

\[ \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 4 \\ 1 & -3 \end{pmatrix} \quad V^{-1} = \begin{pmatrix} \frac{3}{7} & \frac{4}{7} \\ -\frac{1}{7} & -\frac{1}{7} \end{pmatrix} \]

The \(n\)'th order transition matrix \(P^n\) is:

\[ P^n = (VA^{-1})^n = VA^nV^{-1} = V \begin{pmatrix} 1 & 0 \\ 0 & 0.3^n \end{pmatrix} V^{-1} \]

So \(P(X_n = B|X_0 = A) = P^n(1, 2) = \frac{4}{7}(1 - 0.3^n)\)

c.4 Let \(X_T\) be the total money we pay at time \(T\). Let \(T_A\) be the number of total times we stay at \(A\) at time \(T\), \(T_B\) be the number of total times we stay at \(B\) at time \(T\). Then \(T_A + T_B = T\) and \(X_T = T_A + 2T_B\).

\[
E(X_T) = E(T_A + 2T_B) \\
= T + E(T_B) \\
= T + \sum_{n=1}^{T} P(X_n = B|X_1 = A) \\
= T + \sum_{n=1}^{T} \frac{4}{7}(1 - 0.3^{n-1}) \\
= \frac{11T}{7} + \frac{40}{49}(1 - 0.3^T)
\]

**Problem 4 Rolling Fair Dice**
a. You can bound the probability with the Chernoff bound, Markov inequality or the Chebyshev inequality. I will bound it using Chebyshev inequality. Write $Y_n = \sum_{i=1}^n X_i$. Then $E(Y_n) = E(X_1) = 3.5$, $\text{Var}(Y_n) = \frac{\text{Var}(X_1)}{n} = \frac{35}{12n}$. So the Chebyshev inequality gives:

$$P(Y_n \geq 5) \leq P(|Y_n - 3.5| \geq 1.5) \leq \frac{\text{Var}(Y_n)}{1.5^2} = \frac{35}{27n}$$

b. When $n$ is large, we can use the Central Limit Theorem: the CDF of $\frac{\sum_{i=1}^n X_i - 3.5}{\sqrt{\frac{12n}{35}}} \text{ converges to the standard normal CDF. Let } N \text{ be a standard normal random variable. So approximately:}$

$$P\left(\frac{\sum_{i=1}^n X_i}{n} \in [3.5, 3.6]\right) = P\left(\frac{\sum_{i=1}^n X_i - 3.5}{\sqrt{\frac{12n}{35}}} \in \left[0, 0.1 \times \sqrt{\frac{12}{35}}\right]\right)$$

$$= P(N \in \left[0, \sqrt{\frac{12n}{3500}}\right])$$

$$= Q\left(\sqrt{\frac{12n}{3500}}\right) - Q(0)$$

$$= Q\left(\sqrt{\frac{12n}{3500}}\right) - \frac{1}{2}$$

c. Write the outcome sequence as $X_1X_2...X_{100}$, 50 of the $X_i$’s are $T$, the rest are $H$. There are $\binom{100}{50}$ such sequences. Among them there are $\binom{96}{50}$ sequences with $X_1X_2X_3X_4 = TTTT$. So the probability that the first 4 flips are all heads is

$$\frac{\binom{96}{50}}{\binom{100}{50}} = \frac{\frac{96!}{50!46!}}{\frac{100!}{50!50!}} = \frac{50 \times 49 \times 48 \times 47}{100 \times 99 \times 98 \times 97} = 0.0587$$