Fall 2005: EECS126 Practice Midterm II (harder than the actual exam)

Problem 1 *True or False. Prove or show a counterexample:*

a. *If X is uniformly distributed in $[0,1]$, then $\forall$ CDF F, $\exists g$, s.t. random variable $Y = g(X)$ has CDF F.*

b. *If X, Y are jointly continuous with pdf f_{XY} and the level sets $S_k = \{(x,y)|f_{XY}(x,y) = k\}$ are circles for all $k \geq 0$, then X, Y are jointly Gaussian.*

c. *If X, Y are two jointly continuous random variables, write the MMSE estimate of X given Y as $\hat{X}$, and the estimation error $\tilde{X} = X - \hat{X}$. Then*

$$Var(X) = Var(\hat{X}) + Var(\tilde{X})$$

Problem 2 *Poisson Processes*

a. *Let $Y = X_1 + \cdots + X_N$, where the random varbiables X_i are exponential with parameter λ, and N is geometric with parameter p. Assume that the random variables N, X_1, $X_2, \ldots$, are independent. What is the pdf of Y? (Hint: think about a Poisson random process splitting with probability p)*

b. *Suppose that there was 1 arrivals of a Poisson random process in $[0,1]$. What is the conditional pdf for that arrival time?*

c. *Suppose that there were 2 arrivals of a Poisson random process in $[0,1]$. What is the conditional pdf for the first arrival time? (Hint: think about splitting this Poisson process and then using part (b)...)*

Problem 3 *"More fun with Gaussians"*

Consider the set of random variables constructed by $X_{t+1} = \frac{1}{2}X_t + V_t$ where V_t are i.i.d. unit variance zero mean Gaussian random variables independent of X_0 and the set of random variables defined by $Y_t = X_t + W_t$ where W_t are also i.i.d. standard Gaussian random variables independent from V_t and X_0.

a. *What distribution for X_0 makes the X_t identically distributed?*

b. *If you observe Y_0, what is the best mean squared estimate $\hat{X}_0$ for X_0?*
What is the distribution for the estimation error $\tilde{X}_0 = (X_0 - \hat{X}_0)$?

c. *If you observe Y_0 and Y_1, what is the best mean squared estimate for X_1?*
What is the distribution for the estimation error $\tilde{X}_1 = (X_1 - \hat{X}_1)$?

d. *If instead of remembering Y_0, suppose that instead you just kept $\hat{X}_0$ around and then you observed Y_1. Then, you looked at the difference Z_1 between what you had expected to see ($\frac{1}{2}\hat{X}_0$) and what you actually saw: $Z_1 = (Y_1 - \frac{1}{2}\hat{X}_0)$.*

   Use your answers to (b) and (c) above to express $\hat{X}_1$ in terms of $\hat{X}_0$ and $Z_1 = (Y_1 - \frac{1}{2}\hat{X}_0)$.

e. *Now suppose that at time t, you have access to an optimal estimate $\hat{X}_{t-1}$ based only on past observations that has a Gaussian estimation error $\tilde{X}_{t-1} = (X_{t-1} - \hat{X}_{t-1})$ with variance e_{t-1} and zero mean.*[1]

   Give the best mean squared estimator $\hat{X}_t$ based on seeing $\hat{X}_{t-1}$ and the "new" part of the observation Y_t, namely $Z_t = (Y_t - \frac{1}{2}\hat{X}_{t-1})$.

   What is the distribution for the resulting estimation error $\tilde{X}_t = (X_t - \hat{X}_t)$?

f. *Suppose that the past optimal estimate $\hat{X}_{t-1}$ was based on all the observations $Y_0, Y_1, \ldots, Y_{t-1}$.*

   Show that the new estimate you computed in part (e) above has estimation error $\tilde{X}_t$ that is independent of all the observations up through time t: $Y_0, Y_1, \ldots, Y_{t-1}, Y_t$.

   What does that tell you about the estimator we have just constructed?

[1] As is usually the case with optimal estimators in the Gaussian context, the estimation error is independent of the estimate as well as independent of all of the individual observations that were used to generate the estimate.