Fall 2005: EECS126 Midterm 2 on November 10, 2005

No Collaboration Permitted. One sheet of notes is permitted. Turn it in with your exam.

Be clear and precise in your answers

Write your name and student ID number on every sheet.

Come to the front if you have a question.

Problem 1 *(36pts) True or False. Prove or show a counterexample:*

a. *12pts. If $X \sim N(0,1)$, $Y \sim N(0,1)$ and $E(XY) = 0$, then X and Y are independent.*

b. *12pts. If a random variable X has CDF as shown in Fig 1, then there exists a function g, s.t. the random variable $Y = g(X)$ is uniformly distributed on the real interval $[0,1]$.*

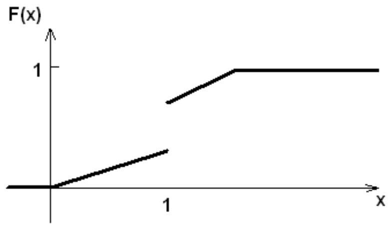


Figure 1: CDF of X

More space:

c. 12pts *Suppose X and Y are jointly continuous random variables with zero mean and finite variances σ_X^2, σ_Y^2. Write the Minimum Mean Squared-Error (MMSE) estimate of X given Y as $\hat{X}$, and the estimation error $\tilde{X} = X - \hat{X}$. Then $\tilde{X}$ is uncorrelated with the observation Y, i.e. $Cov(\tilde{X}, Y) = 0$.*

Problem 2 *Wireless channels(45pts) An inter-building wireless communication system is built between Soda and Cory. Suppose that having a truck drive through the area mixes up the air and causes the wireless channel to change to a new state, at least until the next truck drives by and mixes it up again. If no truck drives through, the channel remains unchanged. Time is modeled discretely into seconds, and every time the wireless channel changes, it is modeled as taking a value from an independent standard Gaussian random variable. The arrivals of trucks are modeled as being iid across time and represented by a Bernoulli process with parameter p.*

The above model can formally be stated as follows. Given iid Bernoulli random variables B_i, $i = 1, 2, \ldots$ where B_i is a Bernoulli random variable $P(B_i = 1) = p$ and iid Gaussian random variables $N_i \sim N(0, 1)$, $i \geq 0$. Define a new collection of random variables representing the channel state X_i, $i \geq 0$ by setting $X_0 = N_0$ and for $i > 0$:

$$X_i = \begin{cases} X_{i-1} & : \quad B_i = 0 \\ N_i & : \quad B_i = 1 \end{cases} \tag{1}$$

a. *10pts. What is the pdf for X_i? (Hint: condition on $B_1, \ldots B_i$)*

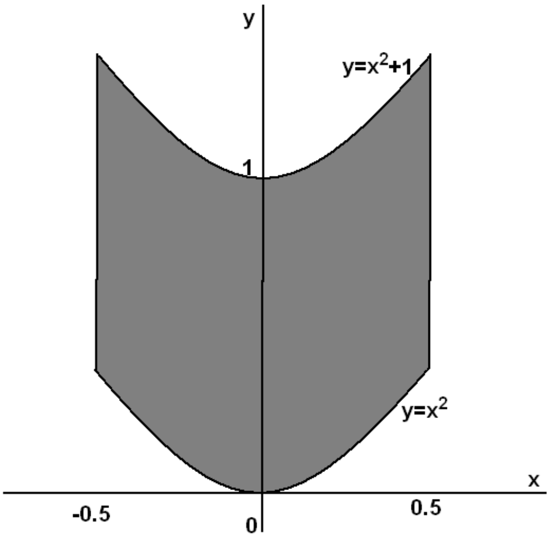
b. *10pts. Are X_1, X_2 jointly continuous, why or why not?*

c. 10pts. What is $Cov(X_1, X_2)$? (Hint: what happens if $B_2 = 0$? what if $B_2 = 1$?)

d. 15pts What is $Cov(X_i, X_j)$, $i < j$*?*

Problem 3 *(40pts) LLSE and MMSE*

Suppose that the random variables X, Y are uniformly distributed in the shaded region bounded by $y = x^2$, $y = x^2 + 1$, $x = -0.5$ and $x = 0.5$, as shown in the following figure.



a. *10pts. Give the MMSE (minimum mean-squared error) estimator for Y given X.*

b. 10pts. Give the LLSE (least linear square estimator) of Y given X.

c. 10pts. Quadratic estimation has the form

$$\hat{Y} = a_1 + a_2 X + a_3 X^2$$

Find the best (in mean square error sense) quadratic estimator of Y *given* X*, i.e derive* a_1, a_2, a_3 *to minimize the expectation of the square error. (Hint: let* $Z = X^2$ *be a new observation)* *d. 10pts. Suppose that X, Y were jointly Gaussian (bivariate Normal) instead. Show that the best (in the mean square error sense) quadratic estimator of Y given X, $\hat{Y} = a_1 + a_2 X + a_3 X^2$ has $a_3 = 0$.*