EECS 126: Probability and Random Processes

Discussion 13

Note: Please work on the problems before the discussion session.

Problem 1 *Problem 6.17 Steady state convergence Consider a Markov Chain with a single recurrent class, and assume that there exists a time $\bar{n}$ such that*

$$r_{ij}(\bar{n}) > 0$$

for all i and all recurrent j. (This is equivalent to assuming that the class is aperiodic). We wish to show that for any i and j, the limit

$$\lim_{n\to\infty} r_{ij}(\bar{n})$$

exists and does not depend on i. To derive this result, we need to show that the choice of the initial state has no long-term effect. To quantify this effect, we consider two different initial states i and k, and consider two independent Markov chains, X_n and Y_n, with the same transition probabilities and with $X_0 = i$, $Y_0 = k$,Let $T = \min\{n|X_n = Y_n\}$ be the first time that the two chains enter the same state.

a. *Show that there exists positive constants c and $\gamma < 1$ such that*

$$P(T \geq n) \leq c\gamma^n$$

b. *Show that if the states of the two chains became equal by time n , their occupancy probabilities at time n are the same , that is:*

$$P(X_n = j|T \leq n) = P(Y_n = j|T \leq n)$$

c. *Show that $|r_{ij}(n) - r_{kj}(n)| \leq c\gamma^n$ for all i, j, k and n. (Hint: condition on the two events $\{T > n\}$) and $\{T \leq n\}$)*

d. *let $q_j^+(n) = \max_i r_{ij}(n)$ and $q_j^-(n) = \min_i r_{ij}(n)$, show that for all n*

$$q_j^-(n) \leq q_j^-(n+1) \leq q_j^+(n+1) \leq q_j^+(n)$$

e. *Show that the sequence $r_{ij}(n)$ converges to a limit that does not depend on i. (Hint: combine the results of parts (c) and (d) to show that the two sequences $q_j^-(n)$ and $q_j^+(n)$ converges and have the same limit.*