EECS 126: Probability and Random Processes

Problem Set 8 Due on November 1^{st}, 2005 in class

Note: Please submit a photocopy of your work. If you collaborate on the assignment, please list the names of students in your study group.

Problem A

X, V, W are i.i.d. standard Normal random variables (0 mean, and variance 1). Let $Y = X + V$ and $Z = X + V + W$. Compute the best mean-squared estimate of X given an observation of Y alone.

Compute the best mean-squared estimate of X given an observation of Z alone.

Compute the best mean-squared estimate of X given both Y and Z.

Discuss.

(Hint. Try to estimate the estimation error from one stage using the new information from the next random variable...)

Problem B "More fun with Gaussians"

Consider the set of random variables constructed by $X_{t+1} = \frac{1}{2}X_t + V_t$ where V_t are i.i.d. unit variance zero mean Gaussian random variables independent of X_0 and the set of random variables defined by $Y_t = X_t + W_t$ where W_t are also i.i.d. standard Gaussian random variables independent from V_t and X_0.

a. What distribution for X_0 makes the X_t identically distributed?

b. If you observe Y_0, what is the best mean squared estimate $\hat{X}_0$ for X_0?

   What is the distribution for the estimation error $\tilde{X}_0 = (X_0 - \hat{X}_0)$?

c. If you observe Y_0 and Y_1, what is the best mean squared estimate for X_1?

   What is the distribution for the estimation error $\tilde{X}_1 = (X_1 - \hat{X}_1)$?

d. If instead of remembering Y_0, suppose that instead you just kept $\hat{X}_0$ around and then you observed Y_1. Then, you looked at the difference Z_1 between what you had expected to see $(\frac{1}{2}\hat{X}_0)$ and what you actually saw: $Z_1 = (Y_1 - \frac{1}{2}\hat{X}_0)$.

   Use your answers to (b) and (c) above to express $\hat{X}_1$ in terms of $\hat{X}_0$ and $Z_1 = (Y_1 - \frac{1}{2}\hat{X}_0)$.

e. Now suppose that at time t, you have access to an optimal estimate $\hat{X}_{t-1}$ based only on past observations that has a Gaussian estimation error $\tilde{X}_{t-1} = (X_{t-1} - \hat{X}_{t-1})$ with variance e_{t-1} and zero mean. (As is usually the case with optimal estimators in the

Gaussian context, the estimation error is independent of the estimate as well as independent of all of the individual observations that were used to generate the estimate.)

Give the best mean squared estimator $\hat{X}_t$ based on seeing $\hat{X}_{t-1}$ and the "new" part of the observation Y_t, namely $Z_t = (Y_t - \frac{1}{2}\hat{X}_{t-1})$.

What is the distribution for the resulting estimation error $\tilde{X}_t = (X_t - \hat{X}_t)$?

f. Suppose that the past optimal estimate $\hat{X}_{t-1}$ was based on all the observations $Y_0, Y_1, \ldots, Y_{t-1}$.

Show that the new estimate you computed in part (e) above has estimation error $\tilde{X}_t$ that is independent of all the observations up through time t: $Y_0, Y_1, \ldots, Y_{t-1}, Y_t$.

What does that tell you about the estimator we have just constructed?

Problem 31. Show that

$$\text{cov}(X_1 + \cdots X_m, Y_1 + \cdots + Y_n) = \sum_{i=1}^{m} \sum_{i=1}^{n} \text{cov}(X_i Y_j).$$

Problem 32. Let $X_1, \ldots, X_n$ be some random variables and let $c_{ij} = \text{cov}(X_i, X_j)$. Show that for any numbers $a_1, \ldots, a_n$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} a_j \geq 0.$$

Problem 38. In a communication system, the value of a random variable X is transmitted, but what is received (denoted by Y) is the value of X corrupted by some additive noise W; that is, $Y = X + W$. We know the distribution of X and W, and let us assume that these two random variables are independent and have the same PDF. Calculate the least squares estimate of X given Y. What happens if X and W are dependent?

Problem 39. Consider three zero-mean random variables X, Y, and Z, with known variances and covariances. Give a formula for the linear least squares estimator of X based on Y and Z, that is, find a and b that minimize

$$\mathbf{E}\big[(X - aY - bZ)^2\big].$$

For simplicity, assume that Y and Z are uncorrelated.

Problem 41. Let U and V be independent standard normal random variables. Let

$$X = U + V, \qquad Y = U - 2V.$$

(a) Do X and Y have a bivariate normal distribution?

(b) Provide a formula for $\mathbf{E}[X \mid Y]$.

(c) Write down the joint PDF of X and Y.