EECS 126: Probability and Random Processes

Problem Set 5 Due on October 11th, 2005 in class

Note: Please submit a photocopy of your work. If you collaborate on the assignment, please list the names of students in your study group.

Problem 3. Find the PDF, the mean, and the variance of the random variable X with CDF

$$F_X(x) = \begin{cases} 1 - \dfrac{a^3}{x^3}, & \text{if } x \geq a, \\ 0, & \text{if } x < a, \end{cases}$$

where a is a positive constant.

Problem 9. Oscar uses his high-speed modem to connect to the internet. The modem transmits zeros and ones by sending signals -1 and $+1$, respectively. We assume that any given bit has probability p of being a zero. The telephone line introduces additive zero-mean Gaussian (normal) noise with variance σ^2 (so, the receiver at the other end receives a signal which is the sum of the transmitted signal and the channel noise). The value of the noise is assumed to be independent of the encoded signal value.

(a) Let a be a constant between -1 and 1. The receiver at the other end decides that the signal -1 (respectively, $+1$) was transmitted if the value it receives is less (respectively, more) than a. Find a formula for the probability of making an error.

(b) Find a numerical answer for the question of part (a) assuming that $p = 2/5$, $a = 1/2$ and $\sigma^2 = 1/4$.

Problem 12. Dino, the cook, has good days and bad days with equal frequency. On a good day, the time (in hours) it takes Dino to cook a souffle is described by the PDF

$$f_G(g) = \begin{cases} 2, & \text{if } 1/2 < g \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

but on a bad day, the time it takes is described by the PDF

$$f_B(b) = \begin{cases} 1, & \text{if } 1/2 < b \leq 3/2, \\ 0, & \text{otherwise}. \end{cases}$$

Find the conditional probability that today was a bad day, given that it took Dino less than three quarters of an hour to cook a souffle?

Problem 15. A family has three children, A, B, and C, of height X_1, X_2, X_3, respectively. If X_1, X_2, X_3 are independent and identically distributed continuous random variables, evaluate the following probabilities:

(a) $\mathbf{P}(A \text{ is the tallest child})$.

(b) $\mathbf{P}(A \text{ is taller than } B \mid A \text{ is taller than } C)$.

(c) $\mathbf{P}(A \text{ is taller than } B \mid B \text{ is taller than } C)$.

(d) $\mathbf{P}(A \text{ is taller than } B \mid A \text{ is shorter than } C)$.

(e) $\mathbf{P}(A \text{ is taller than } B \mid B \text{ is shorter than } C)$.

Problem 17. Let X and Y be independent random variables, with each one uniformly distributed in the interval $[0, 1]$. Find the probability of each of the following events.

(a) $X > 6/10$.

(b) $Y < X$.

(c) $X + Y \leq 3/10$.

(d) $\max\{X, Y\} \geq 1/3$.

(e) $XY \leq 1/4$.

Problem 20. Your driving time to work is between 30 and 45 minutes if the day is sunny, and between 40 and 60 minutes if the day is rainy, with all times being equally likely in each case. Assume that a day is sunny with probability 2/3 and rainy with probability 1/3.

(a) Find the PDF, the mean, and the variance of your driving time.

(b) On a given day your driving time was 45 minutes. What is the probability that this particular day was rainy?

(c) Your distance to work is 20 miles. What is the PDF, the mean, and the variance of your average speed (driving distance over driving time)?

Problem 21. The random variables X and Y have the joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 2, & \text{if } x > 0 \text{ and } y > 0 \text{ and } x + y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let A be the event $\{Y \leq 0.5\}$ and let B be the event $\{Y > X\}$.

(a) Calculate $\mathbf{P}(B \mid A)$.

(b) Calculate $f_{X|Y}(x \mid 0.5)$. Calculate also the conditional expectation and the conditional variance of X, given that $Y = 0.5$.

(c) Calculate $f_{X|B}(x)$.

(d) Calculate $\mathbf{E}[XY]$.

(e) Calculate the PDF of Y/X.