Derived PDF

10/20/04

R.U. X. $\longrightarrow$ Known pdf.

$y = g(x)$

Compute pdf of $g(x)$ y in Terms of pdf of X.

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2 step process.

(1) Find prob event $g \leq g_0$ $\forall g_0$.

$\rightarrow$ CDF for g.

(2) Differentiate CDF in step 1 to get pdf. $\Rightarrow$ Differentiate w.r.t g_0

To get $f_g(g_0)$

Step ①: $y = g(x)$.

Find $F_Y(y) = P(g(x) < y)$

CDF

$$= \int_{\{x \mid g(x) < y\}} f_X(x)\, dx.$$

Step ②: $f_Y(y) = \dfrac{\partial F_Y(y)}{\partial y}$

pdf

Ex wheel of Fortune. $0.00 \longrightarrow 1.00$

spin twice.

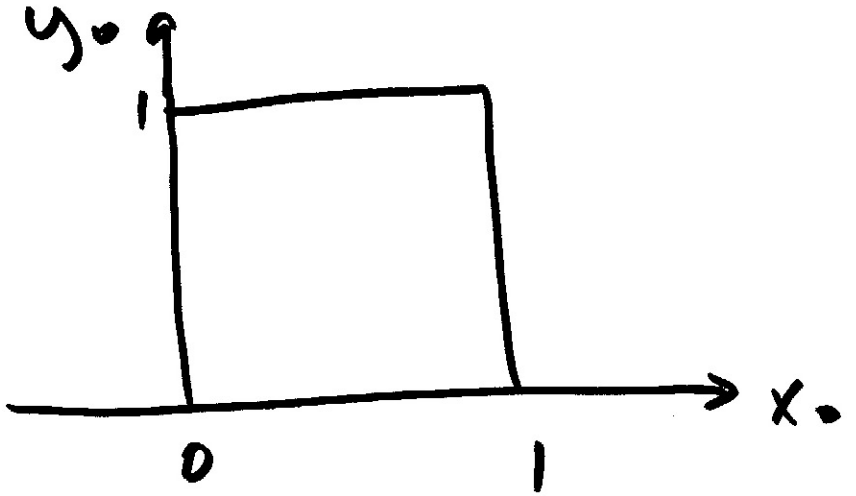
x = first reading

y = 2nd reading.

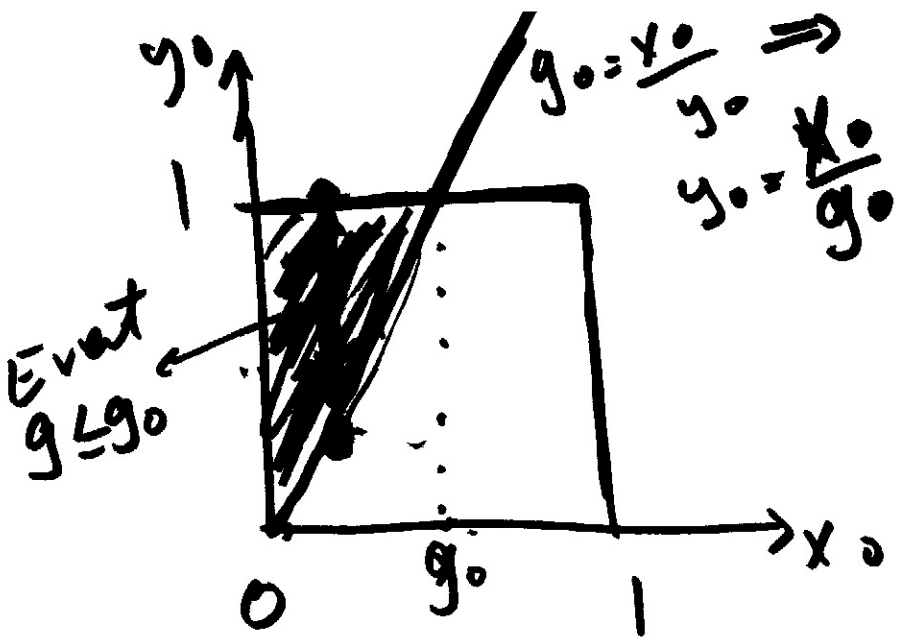
2 spins are independent.

Q pdf $g(x,y) = \frac{x}{y}$ compute $f_g(g_0)$

$$f_{x,y}(x_0,y_0) = f_x(x_0)\, f_y(y_0) = \begin{cases} 1 & 0 \le x_0 \le 1 \\ & 0 \le y_0 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$



$P_{g_\le}(g_0)$ ..



Case ① $0 \le g_0 \le 1$

Case ② $1 \le g_0 \le \infty$

Case ①: $P_{g\le}(g_0) = \int_0^{g_0} dx_0 \int_{y_0 = \frac{x_0}{g_0}}^{1} 1 \; dy_0$

$$\boxed{P_{g\le}(g_0) = \frac{g_0}{2}} \leftarrow \text{CDF}$$

Case (2): $1 \le g_0 \le \infty$

$$P_{g \le g_0} = 1 - \text{unshaded area} = \text{shaded area}.$$

$$= 1 - \int_{x_0=0}^{1} dx_0 \int_{0}^{\frac{g_0}{x_0}} 1 \, dy_0$$

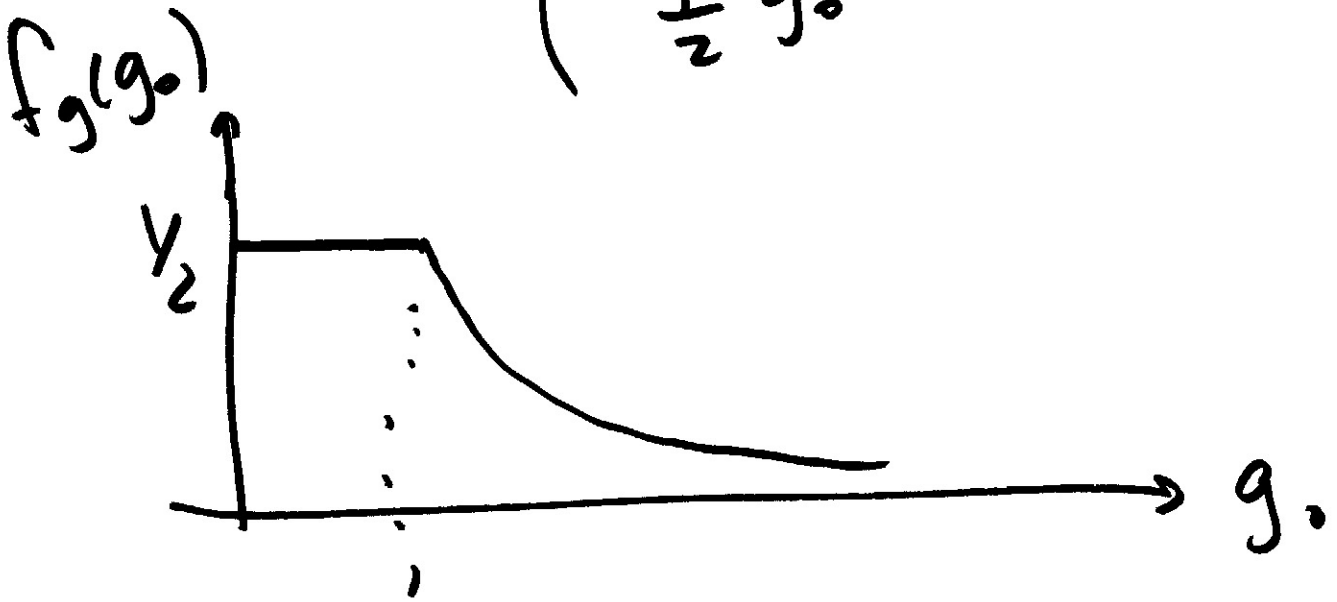
$$\boxed{P_{g \le}(g_0) = 1 - \frac{1}{2g_0}} \leftarrow \text{CDF}.$$

$$P_{g \le}(g_0) = \begin{cases} 0 & g_0 \le 0 \\ \frac{g_0}{2} & 0 < g_0 \le 1 \\ 1 - \frac{1}{2g_0} & 1 \le g_0 \le \infty \end{cases}$$

CDF

Differentiate to get pdf:

$$f_g(g_0) = \begin{cases} 0 & g_0 \le 0 \\ \frac{1}{2} & 0 < g_0 \le 1 \\ \frac{1}{2} g_0^{-2} & 1 \le g_0 < \infty \end{cases}$$



Ex $y = g(x) = x^2$

x is a cont. R.V. with known PdF.

Q what is Pdf for y?

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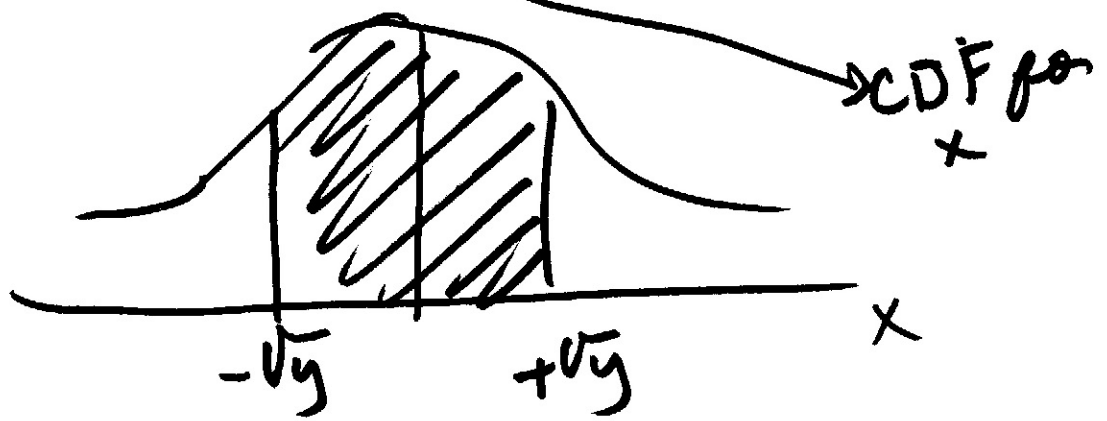
① Find $F_Y(y) = P(g(x) < y) = \int_{\{x \mid g(x) < y\}} f_X(x)\,dx$

$F_Y(y) = P(Y < y)$

$= P(X^2 < y)$

$= P(-\sqrt{y} \le X < \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$



$$\frac{dF_Y(y)}{dy} = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

Pdf for X

Pdf of a Linear fn of a Cont. R.V.

X is cont. R.V. Pdf f_X → known.

$Y = aX + b$ $\quad a \neq 0$ $\quad a, b$ constant.

Q what is $\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$

Proof assume $a > 0$ without loss of generality

CDF for Y

$$F_Y(y) = P_a(Y \le y)$$

$$= P(aX + b \le y)$$

$$= P\left(X < \frac{y-b}{a}\right) \quad a > 0$$

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right)$$

Now Diff to get pdf.

$$\frac{dF_Y(y)}{dy} = \frac{d}{dy}\left\{F_X\left(\frac{y-b}{a}\right)\right\} \leftarrow \text{Chain rule.}$$

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \qquad a > 0$$

Show similarly for $a < 0$

EX X is normal mean μ, var σ^2

$Y = aX + b \qquad a \neq 0$ what is pdf Y?

Q $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) - \frac{\left(\frac{y-b}{a} - \mu\right)^2}{2\sigma^2}$

$f_Y(y) = \frac{1}{|a|} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-b-a\mu)^2}{2\sigma^2 a^2}}$

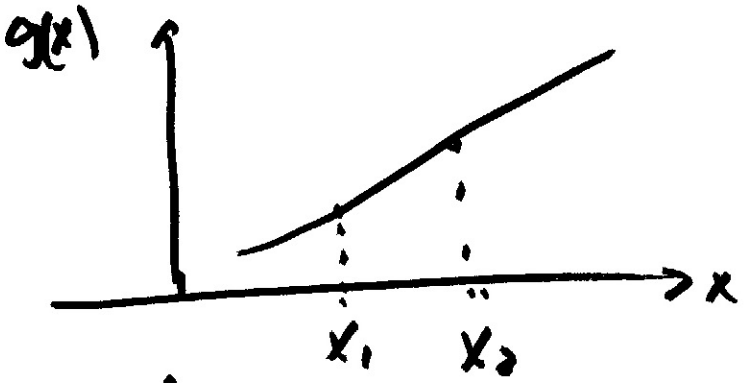
$f_Y(y) = \frac{1}{\sqrt{2\pi}\,|a|\sigma} e^{}$

normal with mean $a\mu + b$, var $a^2\sigma^2$

PDF for a strictly monotic fn of a cont. R.V. X.

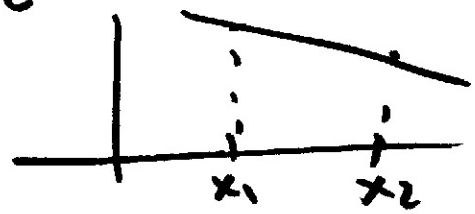
Def strictly monotically increasing fn g over an interval I is defined.

$$g(x) < g(x') \quad \forall x, x' \in I \quad \text{s.t.} \quad x < x'$$



Def monotonically decreasing. fn g over an interval I is defined as:

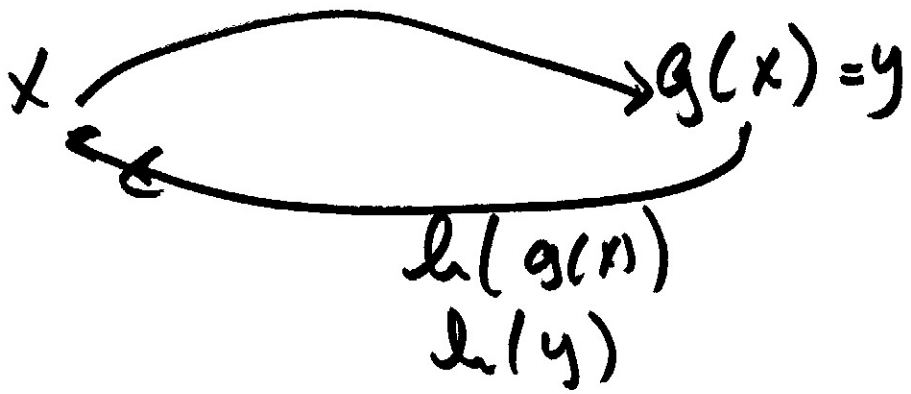
$$g(x) \not> g(x') \quad \forall x, x' \in I \quad \text{s.t.} \quad x < x'$$



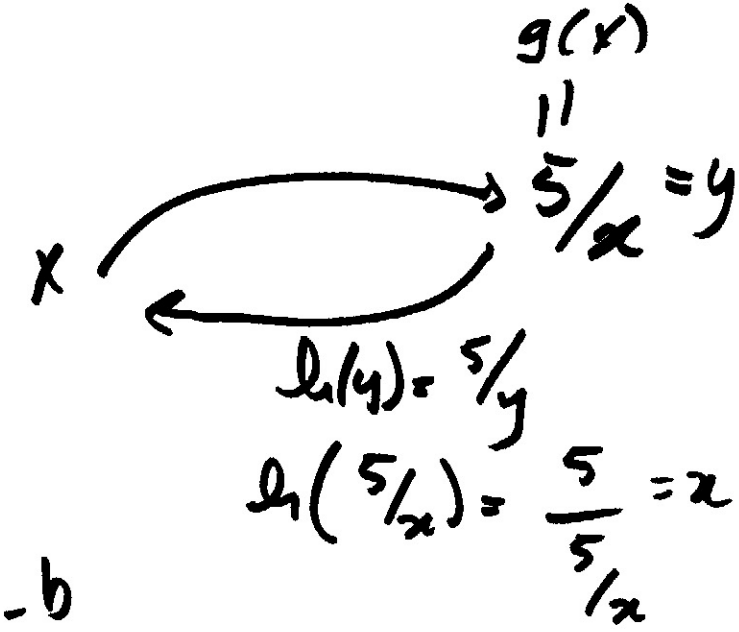
Factoid: if g strictly monotonic $\exists\ h$

i.e. inverse. s.t. $x \in I$

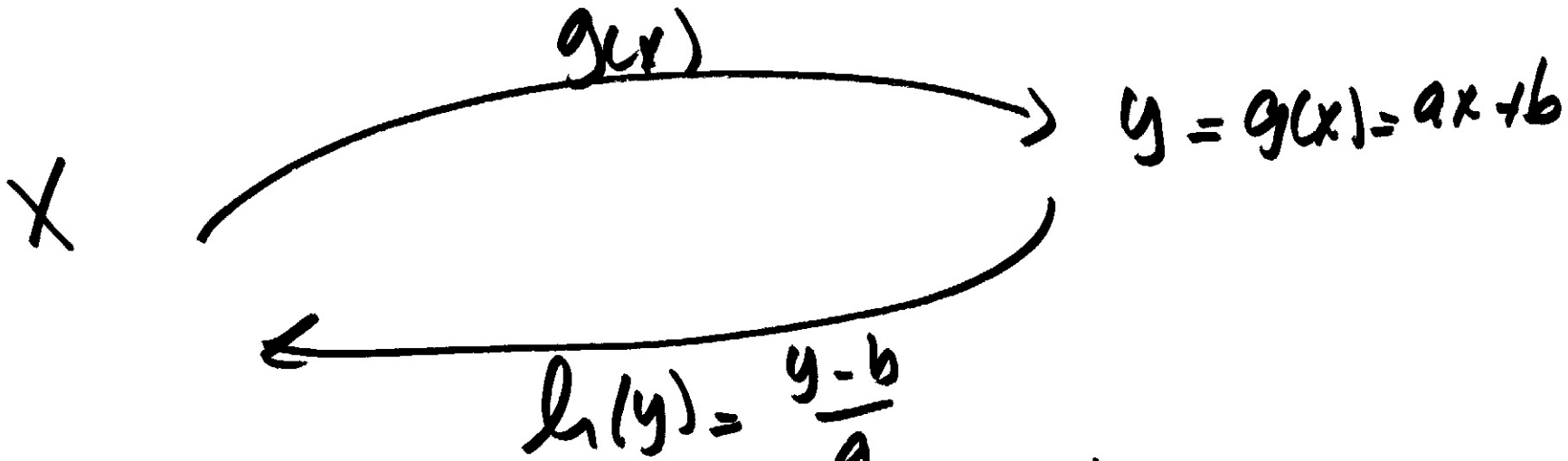
$$y = g(x) \iff x = h(y)$$



Ex $g(x) = 5/x$



Ex $g(x) = ax + b \rightarrow h(y) = \dfrac{y-b}{a}$



$$= h(ax+b)$$

$$= \frac{(ax+b)-b}{a}$$

$$= \frac{ax}{a} = x$$

pdf for strictly monotonic fn of a Cont. R.V. X

R.V. X, known pdf. f_X.

g strictly monotonic. with inverse h over range of X

$$y = g(x) \iff x = h(y)$$

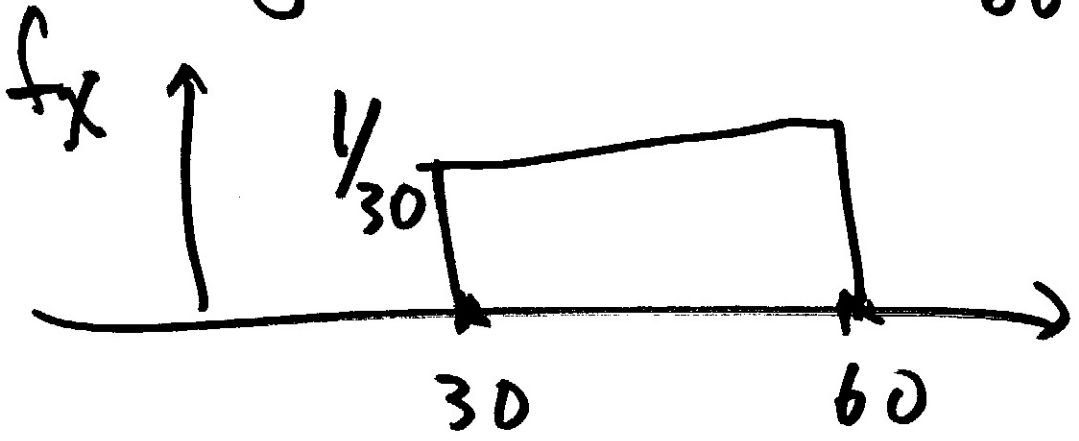
Assume h is differentiable.

Then pdf of Y in the region $f_Y(y) > 0$ is.

$$f_Y(y) = f_X(h(y)) \left| \frac{dh(y)}{dy} \right|$$

Ex X = Speed of vehicle.

Uniformly dist. between 30 MPH
60 MPH



SF → Carmel = 180 Miles.

Pdf of Time it Takes to get to Carmel?

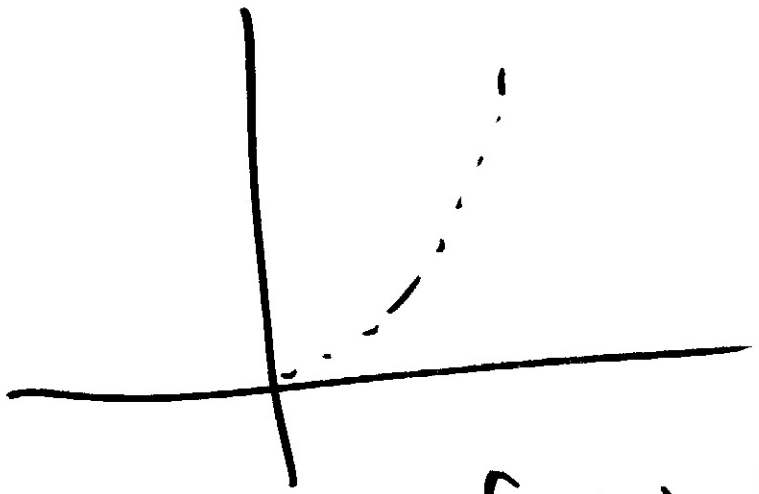
$$y = \frac{180}{X} = g(x) \qquad h(y) = \frac{180}{y}.$$

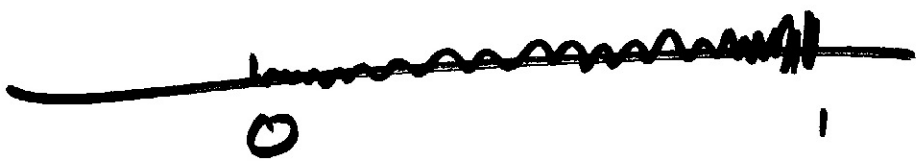
$$f_Y(y) = \frac{1}{30}\left|\frac{d}{dy}\left(\frac{180}{y}\right)\right| = \frac{1}{30}\,\frac{180}{y^2} = \begin{cases} \frac{6}{y^2} & 30 \le y \le 60 \\ 0 & \text{otw} \end{cases}$$

Ex X Uniform R.V. $(0,1]$ → includes 1

everything except 0

$g(x) = x^2$





$h(y) = \sqrt{y} \quad \forall\, y \in (0,1]$

$$f_Y(y) = \begin{cases} 1 \cdot \left| \frac{dh(y)}{dy} \right| = \frac{1}{2\sqrt{y}} & y \in (0,1] \\ 0 & \text{otherwise} \end{cases}$$

2 R.V. g. h.

CDF ① $P_{g\le, h\le}(g_0, h_0)$ $\forall g_0, h_0$

② $f_{g,h}(g_0, h_0) = \dfrac{\partial^2 P_{g\le h\le}(g_0, h_0)}{\partial g_0 \partial h_0}$

Ex Y, X uniformly dist. RV. [0,1].
independent.

pdf of $Z = \dfrac{Y}{X}$ = [redacted] $g(x,y)$.

CDF